On the Non-Existence of Reputation Effects in Two-Person Infinitely-Repeated Games

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Abstract

Consider a two-person infinitely-repeated game in which one player is either a normal "rational" type or a "commitment" type that automatically plays a fixed repeated-game strategy. When her true type is private information, a rational type may want to develop a reputation as a commitment type by mimicking the commitment type's actions. But, the uninformed player, anticipating the behavior of the rational type, may try to "screen out" the rational type by choosing an action which gives the rational type a low payoff when she mimics the commitment type. My main result shows that for "comparably" patient players, if the prior probability that the player is a commitment type is sufficiently small, the "screening" process may take so long that the rational type does not benefit from developing a reputation. In the case of equally patient players, I show that the folk theorem holds even when both players possess a small amount of private information. Schmidt (1994) and Cripps, Schmidt and Thomas (1993) argue that reputation effects can rule out outcomes permitted by the folk theorem, regardless of how small the prior probability that the player is a commitment type. My results show that this argument only applies when one player is "infinitely" more patient than the other.

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1 Introduction

In this essay, I study the effects of reputation in a class of incomplete-information, two-person, infinitely-repeated games. They differ from standard perfect-information repeated games in that one of the players possesses private information as to whether she is a rational or "commitment" type who plays a fixed repeated-game strategy. Because he does not know the true type of the informed player, the uninformed player can only form a belief about the informed player's type based on her past actions. As a result, a rational informed player can try to develop a "reputation" as a commitment type by mimicking her behavior. The goal is to study whether the possession of a small amount of private information allows the informed player to obtain a higher payoff than in the case of perfect information.

Most previous work in the literature consider the case where the uninformed player's discount factor is fixed as the informed player's approaches one. As the informed player's discount factor increases, she would become relatively, and in the limit *infinitely,* more patient than the uninformed player. Fudenberg and Levine (1989) show that in a game in which a long-run player plays against a sequence of short-run players, if the long-run player is patient enough, she will obtain a payoff close to or better than her Stackelberg payoff, which generally is strictly higher than her minmax payoff. Cripps, Schmidt and Thomas (1993) extend the results of Fudenberg and Levine (1989) to the case of two long-run players. They show that, in general, as the informed player becomes sufficiently more patient than her opponent, she can guarantee herself a payoff strictly higher than her minmax payoff. Under some conditions, she can guarantee herself a payoff close to her Stackelberg payoff (Schmidt 1993), and in some others, close to her highest repeated-game payoff (Celentani, Fudenberg, Levine, and Pesendorfer 1993).

The condition of infinite relative patience is obviously highly restrictive. Many economic relationships, for example, repeated oligopolistic competition, involve parties that are equally or, at least, comparably patient. In this paper, I study reputation effects when the informed player is only finitely more patient than the uninformed player in the limit, or, put differently, where the players are equally or *comparably* patient. My main result shows that in any two-person, infinitely-repeated game, except for two special classes specified below, if the two players are equally and sufficiently patient, the commitment strategy is history-independent, and the prior probability of that the informed player is a commitment type is sufficiently small, then any strictly individually rational payoff profile can be supported by a perfect Bayesian equilibrium. Under slightly stronger conditions, the same result applies even when the commitment strategy is history-dependent, the players are comparably (but not equally) patient, and both players possess private information. In other words, the minimum equilibrium payoffs that the informed player may receive with or without private information are the same. In conclusion, reputation effects do not exist when the players are comparably patient, and the prior probability that the informed player is a commitment type is sufficiently small.

Compared to previous results, my results suggest that the strength of reputation effects critically depends on the players' relative patience. For any fixed prior probability that a player is a commitment type, reputation effects become stronger as the informed player becomes more patient than her opponent. But, when the relative patience between players is fixed, reputation effects deminish as the prior probability that the player is a commitment type decreases. Call the informed player Player 1 and the uninformed player Player 2. While Player 2 is uncertain about Player 1's type, in equilibrium, he should expect that a rational Player 1 will mimic the commitment type and, hence, will not quickly conclude that his opponent is a commitment type after observing her choosing the commitment strategy. Instead, Player 2 may choose an action which will give the rational type of Player 1 a low payoff when she chooses the commitment strategy so as to "screen out" the rational type. "Screening" is rational for Player 2 if he believes that the rational type of Player 1 may reveal her type in the future, and if in this case he will obtain a higher payoff in the continuation game. Suppose T is the maximum number of screening periods that is consistent with some equilibrium.[1] If Player 1 plays the commitment strategy indefinitely, she will receive a low average payoff v_1^s during the screening periods, and a high average payoff v_1^* after the screening periods, when Player 2 is convinced that she is a commitment type. Player 1's average discounted payoff will be equal to

$$(1 - \delta_1^T)v_1^s + \delta_1^T v_1^*,$$

where δ_1 is Player 1's discount factor. If δ_1^T is small, her average payoff is approximately equal to the low payoff during the screening periods, and there is little to gain from developing a reputation. When Player 2 is more patient, he is willing to screen for more periods; therefore, T is an increasing function of δ_2, Player 2's discount factor. Notice that the cost of screening to Player 2 is at most equal to $(1 - \delta_2^T)d$ where d is equal to the largest difference in payoffs between any two stage-game outcomes. Player 2 is willing to pay some positive cost to screen whenever he expects to receive some positive long-run benefit in the event that the screening is successful; therefore, δ_2^T should be

[1] T is finite because Player 2 will screen in a period only if he expects the rational type of Player 1 will reveal her type with a probability bounded away from zero. See Section 2 for details.

less than 1 in the limit. Furthermore, the limit of δ_2^T will be small when it is unlikely that Player 1 is a commitment type. The relative patience of the players matters because it determines the relative sizes of δ_1^T and δ_2^T. When Player 1 and Player 2 are comparably patient in the limit, if the limit of δ_2^T is small, then the limit of δ_1^T will be small as well. This explains why the results of Cripps, Schmidt and Thomas (1993) and others do not apply to this case.

There are two classes of games which are exceptions to the above argument. In both cases, there must exist a stage-game action a_1 for Player 1 such that if Player 1 chooses a_1 and Player 2 chooses a best response to it, then Player 1 receives her highest payoff in the convex hull of the set of strictly individually rational stage-game payoffs. In this case, Player 1 will obtain her highest repeated-game payoff if she can credibly commit to choosing a_1. A stage game is called a *strongly-conflicting-interest game* if Player 2 will get his minmax payoff when he best responds to a_1. A stage game is called a *strongly-dominant-action game*, and a_1 is called a *strongly-dominant action*, if a_1 is also a strictly dominant action for Player 1.[2] Notice that the argument above critically depended on the facts (1) that the rational type of Player 1 can be induced to reveal her type and (2) that Player 2 will suffer a long-term loss for not screening. A strongly-dominant-action game violates the first condition, while a strongly-conflicting-interest game violates the second condition.[3]

In section 5, I show that in an infinitely-repeated strongly-dominant-action game with one-sided incomplete information, if the only commitment type of the informed player (Player 1) is one who always chooses the strongly-dominant action, then Player 1 will receive her commitment payoff in any perfect Bayesian equilibrium. Notice that in the perfect-information version of the game, there usually exists a large set of equilibrium outcomes. In this case, reputation effects select a unique outcome (the one most favorable to Player 1) from that set. Moreover, unlike all previous results on reputation effects in infinitely-repeated games, which critically depend on the requirement of infinite relative patience in the limit, this result holds for any strictly positive prior probability of a commitment type, and any discount factors less than 1. In fact, it applies even when Player 1 is less patient than Player 2. The basic argument is as follows: Suppose in some equilibrium Player 2 chooses to screen in some period; then he must believe that the rational type of Player 1 will reveal her type with a finite probability in the future. But in strongly-dominant-action games, the rational type of Player 1 will reveal her type only when she expects Player 2 to screen in the future. Hence, by repeating the same argument, we can conclude that in such an equilibrium Player 2 has to choose to screen in an infinite number of periods. The key of the proof is to show that

[2]These terms are formally defined in section 4.

[3]I would like to thank Eddie Dekel and Wolfgang Pesendorfer for pointing out the second point to me.

this cannot happen because eventually Player 2 will be convinced that Player 1 is a commitment type. Hence, any screening by Player 2 is not consistent with a perfect Bayesian equilibrium and, thus, the rational type of Player 1 must receive her commitment payoff in any perfect Bayesian equilibrium. Notice that this argument is different from the one introduced by Fudenberg and Levine (1989) and commonly used in the reputation literature. That argument only requires that Player 2 update his beliefs rationally and behave optimally given his beliefs, while my argument, in addition, also makes use of the fact that the strategy of the rational type of Player 1 is a best response to Player 2's strategy in equilibrium.

There are two recent papers which address issues similar to those in this one.[4] Cripps and Thomas (1997) demonstrate that reputation effects do not exist in a class of two-person infinitely-repeated common-interest games with equally patient players. They show that when the players are sufficiently patient, and one player may be a commitment type who always plays the Pareto-dominant action, there exists a perfect Bayesian equilibrium in which the payoff for the informed player is close to her minmax payoff. The general idea of their proof is similar to that of Theorem 1, but their result only applies when the stage game belongs to a special class of common-interest games. Cripps (1997) studies infinitely-repeated games with one-sided incomplete information. He shows that when uncertainty is small and the players are sufficiently patient, any strictly individually-rational payoffs can be supported by a perfect Bayesian equilibrium. Despite apparent similarities, his model is not about reputation effects. Unlike other recent work on this topic, including this one, which assumes that the commitment type plays a fixed commitment strategy, Cripps assumes that the commitment type possesses a stage-game payoff function different from that of the rational type.[5] Since he specifically rules out by assumption the possibility that a commitment type will choose a fixed repeated-game strategy, his results do not apply to any of the cases studied in this paper.[6]

The rest of the paper is organized as follows: In Section 2, I introduce a measure of relative

[4] My results are independently of these.

[5] In Cripps' model, all types are rational. I continue to use the term "commitment type" and "rational type" for convenience only.

[6] For example, consider a simple commitment type who chooses a fixed stage-game action independent of the strategy of her opponent. If we want to model a simple commitment type as a rational player who has a different stage-game payoff function, the payoff function must give the commitment type her highest stage-game payoff whenever she chooses the commitment action (regardless of what the other player chooses). If not, when the commitment player is patient enough, there are always some repeated-game strategies for the other player which will induce the commitment player to deviate from the commitment action. This implies that the minmax payoff for the commitment type is equal to her highest stage-game payoff, and the set of her individually-rational payoffs is a singleton. Assumption A.1 in Cripps (97) rules out any payoff functions with this property. This explains why the two exceptions discussed in this paper do not appear in his model.

patience, and apply it to study the relationship between relative patience and reputation effects. This section makes clear that the qualitative results which hold in the the simple case of equally patient players also apply to the more general case of comparably patient players. In Sections 3, 4, and 6, I assume the players are equally patient. But under a minor technical assumption, the results in these sections apply to the case of comparably patient players as well. Section 3 introduces a model of two-person infinitely-repeated games with one-sided incomplete information. In Section 4, I establish the main result of this paper: Under fairly general conditions, when the prior probability that the informed player is a commitment type is sufficiently small, there exists a perfect Bayesian equilibrium in which the payoff for the informed player is close to her minmax payoff. Section 5 considers an important exception to the general result that reputation effects do not exist in two-person infinitely-repeated games. I show that in the case of infinitely-repeated strongly-dominant-action games, if there is a slight probability that a player is a commitment type who always chooses the strictly-dominant action, then that player will receive her commitment payoff in any perfect Bayesian equilibrium. In Section 6, I extend the result in Section 4 and prove a folk theorem with two-sided incomplete information. Section 7 contains the conclusion.

2 Reputation Effects and Relative Patience

Absolute patience and relative patience are two distinct concepts. A player can be a lot more patient than another player, even when both are very patient. Consider any strictly increasing, continuously differentiable function $\delta_2(\delta_1)$, where $\delta_2 : [0,1] \to [0,1]$ such that $\lim_{\delta_1 \to 1} \delta_2(\delta_1) = 1$. This function $\delta_2(\delta_1)$ expresses Player 2's discount factor as a function of Player 1's. The graph $\{\delta_1, \delta_2(\delta_1)\}_{\delta_1=0}^{1}$ defines a path ending at $(1,1)$ on the unit square. As the discount factors move along the path toward $(1,1)$, the players become more patient. Conventional folk theorems, assuming that the players are equally patient, characterize the limiting set of equilibrium payoffs as the common discount factor approaches one along the diagonal. But in general, the discount factors of the players need not be the same along any particular path. The relative patience between two players with discount factors δ_1 and δ_2 can be measured by, $m = \frac{\ln \delta_1}{\ln \delta_2}$, the ratio of the logs of their discount factors. Note that m compares the weight the two players put on their average discounted payoffs after any period t. For any t, $\delta_1^t = (\delta_2^t)^m$, where δ_i^t is the weight player i puts on the average payoff after period t. Player 1 is said to be more (less) patient than Player 2 when m is smaller (bigger) than one, since she puts more (less) weight on her future payoff. Definition 1 formalizes the notion of *infinite relative patience.*

Definition 1 *Along any path* $\{\delta_1, \delta_2(\delta_1)\}_{\delta_1=0}^{1}$, *Player 1 is infinitely more patient than Player 2 in the limit if*

$$\lim_{\delta_1 \to 1} m(\delta_1, \delta_2) = 0$$

The players are called *comparably patient* in the limit if neither player is infinitely more patient than the other. Geometrically, the condition of infinite relative patience is equivalent to the condition that the slope of $\delta_2(\delta_1)$ go to infinity as δ_1 goes to 1.

Whether a number of periods is "long" for a player depends on her discount factor. For example, one hundred periods may be "short" for a player with a discount factor of 0.9999 but very "long" for a player with a discount factor of 0.9. This notion of "length" can be measured by δ_i^t; t periods are "long" for Player 1 if δ_1^t is small, because Player 1 cares little about her payoffs after t periods. Lemma 2.1 shows that along any path, a number of periods that is arbitrarily "long" for Player 2 becomes arbitrarily "short" for Player 1 in the limit if and only if Player 1 is infinitely more patient than Player 2. The proofs of all lemmas in this section are given in the appendix.

Lemma 1 *Player 1 is infinitely more patient than Player 2 in the limit as* δ_1 *goes to 1 iff*

$$\forall\, \epsilon, \eta \in (0,1),\ \exists \underline{\delta}_1 \ s.t.\ \forall \delta_1 \geq \underline{\delta}_1,\ and\ \ \forall t \in \Re,\ \delta_2(\delta_1)^t = \epsilon \Rightarrow \delta_1^t \geq\ 1 - \eta. \tag{1}$$

Along any path, both players become more patient. In that case, while Player 1 has a stronger incentive to develop a reputation, Player 2 also has a stronger incentive to screen out the rational type of Player 1. In the limit, whether Player 1 can benefit from developing a reputation depends on the relative patience of the players. The following example illustrates the relationship between relative patience and reputation effects.

		Player 2 D	Player 2 C
Player 1	D	-d,-d	q,0
	C	0,q	q,q

Figure 1: Example 1

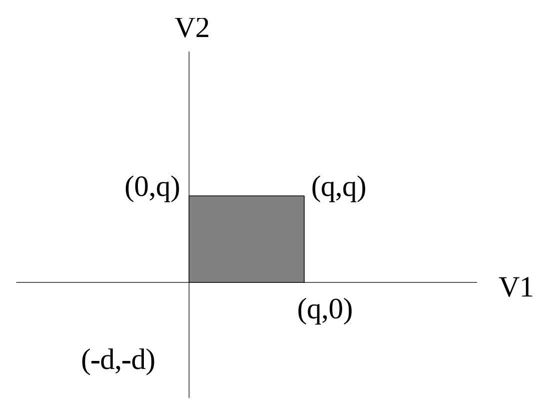

Figure 2: Stage-game payoffs

In the stage game depicted in Example 1, there are three pure strategy equilibria: (C,D), (C,C) and (D,C).[7] The minmax payoffs for both players are 0. The convex hull of the set of feasible stage-game payoffs is shown in Fig. 2. The shaded area is the set of subgame-perfect-equilibrium payoffs of the infinitely-repeated version of this game when the players' discount factors are equal and close to one. Assume that there are two types of Player 1: a rational type with probability $1-\mu_0$, and a commitment type, who always chooses D, with probability μ_0. Let μ_t be Player 2's posterior belief that Player 1 is a commitment type after observing her period $t-1$ action. If Player 2 is convinced that Player 1 is the commitment type, his best response is to play C, and Player 1's corresponding commitment payoff is q.

In equilibrium Player 2 may try to screen out the rational type of Player 1 by choosing D for T periods unless Player 1 chooses C. Screening is optimal for Player 2 only when the rational type of Player 1 will choose C with some positive probability in some periods. Suppose the strategy of the rational type of Player 1 during the screening phase is as follows: In any period t, if she has not yet revealed her type, then with probability p_t, she will do so by choosing C. If she chooses C and Player 2 chooses D, then Player 1 will always choose C from period $t+1$ onward. On the other hand, if Player 2 chooses C that period, Player 1 will always choose D from period $t+1$ onward. Player 2's average continuation payoff will be q in the first case and 0 in the second. Note that Player 2's action in period t will affect his future payoff only when Player 1 chooses C in that period. In choosing D in period t, he pays a short-term cost slightly less than d in that period in return for an expected long-term gain of $p_t(1-\mu_t)q$ in the next period, where $p_t(1-\mu_t)$ is the probability that Player 1 is rational and will reveal her type in that period. Thus, Player 2 will strictly prefer to choose D if

$$(1-\delta_2)d < \delta_2 p_t(1-\mu_t)q \tag{2}$$

To satisfy Equation (2), p_t cannot be too small. Set $p_t = \frac{\Delta}{1-\mu_t}$ where

$$\Delta = \frac{(1-\delta_2)d}{\delta_2 q}. \tag{3}$$

Since the rational type of Player 1 chooses D with a probability strictly less than 1, according to Bayes' rule, Player 2 will assign a higher posterior belief that Player 1 is a commitment type every time after Player 1 chooses D. Specifically, the posterior belief will be equal to:

[7]This example is non-generic, but the conclusion I draw applies in general. I choose this example for brevity. A more general treatment is given in the Sections 3 and 4.

$$
\begin{aligned}
\mu_{t+1} &= \frac{\mu_t}{\mu_t + (1-\mu_t)(1-p_t)} && (4)\\
&= \frac{\mu_t}{1-\Delta} && (5)\\
&= \frac{\mu_1}{(1-\Delta)^t} && (6)
\end{aligned}
$$

Equation (6) shows that μ_t is an increasing function of t . Since μ_{t+1} cannot exceed 1, screening cannot last forever in equilibrium. Intuitively, in order to induce Player 2 to screen, the rational type of Player 1 has to reveal her type with a probability bounded away from zero in every period. This imposes a lower bound on the rate of Bayesian updating and, hence, an upper bound on the number of screening periods.[8] The number of the screening periods, T, is set so that at the end of the screening phase, Player 2's posterior belief that Player 1 is a commitment type is just less than $\bar{\mu} \leq 1$.[9] Formally,

$$T = \max\{t : \frac{\mu_1}{(1-\Delta)^t} \leq \bar{\mu}\}. \tag{7}$$

If a rational type of Player 1 plays the commitment strategy forever, her average payoff, v_1^*, will be equal to

$$v_1^* = (1-\delta_1^T)(-d) + \delta_1^T q.$$

If δ_1^T is close to 1, then v_1^* is approximately equal to q; i.e. Player 1 obtains a payoff close to the commitment payoff by mimicking the commitment type. Basically, δ_1^T measures the "length" of the screening phase relative to Player 1's patience.

Formally, along any path $\{\delta_1, \delta_2(\delta_1)\}_{\delta_1=0}^1$, we can define $\Delta(\delta_1)$, $\mu_t(\delta_1)$, and $T(\delta_1)$ according to Equations (3), (6) and (7). Note that $\Delta(\delta_1)$ is decreasing in δ_1, and goes to 0 as δ_1 goes to 1, i.e. $\lim_{\delta_1 \to 1} \Delta(\delta_1) = 0$. This implies that the rate of belief updating will decrease as the players become more patient. Intuitively, when Player 2 is more patient, he will value his future payoff more than his current payoff; therefore, he can be induced to screen even when the probability that Player 1 will deviate from the commitment strategy is small. As a result, the number of screening periods, T, is increasing in δ_2. The next lemma characterizes the limits of $\delta_1^{T(\delta_1)}$ and $\delta_2^{T(\delta_1)}$ as functions of q,d, μ_0, $\bar{\mu}$.

Lemma 2 *Along any path* $\{\delta_1, \delta_2(\delta_1)\}_{\delta_1=0}^1$,

[8] This argument was introduced by Fudenberg and Levine (1989).

[9] Obviously, $\bar{\mu}$ cannot be bigger than 1. In this particular example, we may simply assume $\bar{\mu} = 1$, but in Section 4.2, when we consider that case of history-dependent commitment strategy, we will need $\bar{\mu}$ to be strictly less than 1.

1. $\lim_{\delta_1 \to 1} \delta_2(\delta_1)^{T(\delta_1)} = \frac{\mu_0}{\bar{\mu}} \frac{q}{d}$.

2. *If* $\lim_{\delta_1 \to 1} m(\delta_1, \delta_2(\delta_1)) = \bar{m} > 0$, *then* $\lim_{\delta_1 \to 1} \delta_1^{T(\delta_1)} = \lim_{\delta_1 \to 1} (\delta_2(\delta_1)^{T(\delta_1)})^{\bar{m}} = \frac{\mu_0}{\bar{\mu}}^{\frac{q\bar{m}}{d}}$.

Lemma 2.2 illustrates how various factors affect the length of the screening phase relative to the patience of the players. The variables q, d, μ_0, and $\bar{\mu}$ jointly determine $\lim_{\delta_1 \to 1} \delta_2(\delta_1)^{T(\delta_1)}$, which, together with $\bar{m}$, in turn determines $\lim_{\delta_1 \to 1} \delta_1^{T(\delta_1)}$. Note that $\lim_{\delta_1 \to 1} \delta_2(\delta_1)^{T(\delta_1)}$ is smaller the smaller the prior belief that Player 1 is a commitment type, the smaller the short-term loss in screening, and the larger the long-term gains. When Player 1 is relatively more patient than Player 2, reputation effects are magnified because $\lim_{\delta_1 \to 1} \delta_1^{T(\delta_1)}$ will then be relatively bigger than $\lim_{\delta_1 \to 1} \delta_2(\delta_1)^{T(\delta_1)}$. When Player 1 is infinitely more patient than Player 2 in the limit, for any fixed $\mu_0 > 0$, $\lim_{\delta_1 \to 1} \delta_1^{T(\delta_1)} = 1$. As a result, she will always receive a payoff arbitrarily close to the commitment payoff, no matter how small the prior probability that Player 1 is a commitment type. However, that is not true when the players are comparably patient in the limit. The next lemma shows that in that case, for any q, d, and $\bar{\mu}$, $\lim_{\delta_1 \to 1} \delta_1^{T(\delta_1)}$ is close to 0 when μ_0 is sufficiently small.

Lemma 3 *If* $\lim_{\delta_1 \to 1} m(\delta_1, \delta_2(\delta_1)) = \bar{m} > 0$, *then* $\forall\ \epsilon > 0\ \exists\ \bar{\mu_0}$ *and* $\exists\ \underline{\delta_1}$, *such that* $\forall\ \mu_0 \ \Box\ \bar{\mu_0}$ *and* $\forall\ \delta_1 \geq \underline{\delta_1}$, $\delta_1^{T(\delta_1)} \ \Box\ \epsilon$.

Note that Lemma 2.3 applies whenever q and $\bar{m}$ is strictly positive.[10] Thus, the case of equally patient players is qualitatively similar to the case of comparably but not equally patient players. In both cases, if the prior probability that the informed player is a commitment type is sufficiently small, it is possible for the rational type of the informed player to reveal her true type in every period with a probability that is (1) large enough so that it is optimal for the uninformed player to choose a non-best-response to the commitment strategy, yet (2) small enough so that screening can potentially last for a long time. In Sections 3, 4 and 6, I assume that the players are equally patient. But, under a technical assumption which I specify in Section 4.3, all of my results apply to the case of comparably patient players as well.

3 A Model

3.1 Preliminaries

Consider a two-person, infinitely-repeated game $\Gamma(A, g, \delta)$ where (A, g) is the stage game and δ is the players' common discount factor. In each period the two players play a simultaneous-move stage game (A, g), where $A = A_1 \times A_2$ denotes the set of finite stage-game actions, and

[10] Of course, for a given ϵ, the threshold $\bar{\mu}_0$ is smaller the smaller q and $\bar{m}$.

$g = (g_1, g_2) : A \rightarrow \Re^2$ denotes their payoff functions. A pure action, a mixed action and the set of mixed actions of player i are denoted as a_i, α_i and $\mathcal{A}_i$, respectively. Let $br_j(a_i)$ be the set of best responses for Player j to Player i's action a_i. For convenience, I represent the expected payoff of a mixed action profile, $E_\alpha[g(a)]$, by $g(\alpha) = (g_1(\alpha), g_2(\alpha))$, where $\alpha = (\alpha_1, \alpha_2)$. The stage-game payoffs are assumed to be bounded from above and below. Let d be the maximum difference in payoffs between two different outcomes for both players. Let V be the convex hull of the set of feasible stage-game payoffs, and V^* be the convex hull of the set of strictly individually rational stage-game payoff profiles, i.e. $V^* \equiv \{(v_1, v_2) \in V : v_i > \underline{v}_i, i = 1, 2\}$ where $\underline{v}_i$ is the minmax payoff for Player i. Let V_i^* be the projection of V^* on V_i.[11] Let $\bar{v}_i$ denote the highest payoff for Player i in V which is consistent with Player j receiving at least his minmax payoff, i.e. $\bar{v}_1 \equiv \max\{v_1 \in V_1 : \exists\ v_2 \geq \underline{v}_2, \text{ s.t. } (v_1, v_2) \in V\}$. Throughout the paper, I assume that V^* is non-empty, and that the minmax payoffs for the two players, $\underline{v}_1$ and $\underline{v}_2$, belong to the closure of V_i^*. These assumptions simplify the exposition, but are not crucial to my results.

After each period, the players observe their opponent's action, and they have perfect recall. Let the function $h_t^i : \{1, 2.., t-1\} \rightarrow A_i$ represent the actions of Player i up to, but not including, period t. That is, $h_t^i(s)$, for $s > t$, is Player i's period s action. Let $h_t = (h_t^1, h_t^2)$ be a $t-1$ period history of the game and H_t be the set of feasible histories up to period t. Similarly, let h be a complete history and H_∞ be the set of complete histories. Finally, let H be the set of all finite histories of the game. The concatenation of two histories h_t and h_s is represented by $h_t.h_s$. $h_t.h_s$ is a $t+s-2$ period history where $h_t.h_s(r)$ is equal to $h_t(r)$ if $r \leq t-1$ and equal to $h_s(r-t+1)$ if $t \leq r \leq t+s-2$. Similarly, $h_t.a$, for $a \in A$, represents the concatenation of history h_t and the period t action profile a.

The players can condition their actions on past history. A pure repeated-game strategy of Player i is given by a function $s_i : H \rightarrow A_i$; $s_i(h_t)$ is Player i's action in period t after history h_t. Similarly, a mixed repeated-game strategy of player i is given by a function $\sigma_i : H \rightarrow \mathcal{A}_i$, where $\sigma_i(h_t)$ is Player i's mixed stage-game strategy in period t after history h_t. Let S_i and Σ_i be the set of pure and mixed repeated-game strategies of Player i. When it is convenient, I consider σ_i as a probability distribution over the set of pure strategies S_i. Let $\sigma_i|h_t$ denote the continuation strategy of σ_i induced by a history h_t. That is, $\forall\ h_s \in H$, $\sigma_i|h_t(h_s) = \sigma_i(h_t.h_s)$. Let h_{t,s_1,s_2}

[11] $V_1^* \equiv \{v_1 : \exists v_2 \text{ s.t. } (v_1, v_2) \in V^*\}$, and $V_2^* \equiv \{v_2 : \exists v_1 \text{ s.t. } (v_1, v_2) \in V^*\}$.

be the t-1 period history induced by a pure strategy profile (s_1, s_2).[12] A mixed-strategy profile (σ_1, σ_2) induces a probability measure P_{σ_1,σ_2} over H_∞. Denote the support for P_{σ_1,σ_2} by $H(\sigma_1, \sigma_2)$. Denote the set of histories that are consistent with a repeated-game strategy σ_i of Player i by $H(\sigma_i) = \{h_t : \exists \sigma_j, j \neq i \text{ s.t. } P_{\sigma_i,\sigma_j}(h_t) > 0\}$.

Both players try to maximize their discounted expected payoffs. For a history h, the average discounted payoffs of the two players are given by $v(h) = (v_1(h), v_2(h))$, where $v_i(h) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} g_i(h(t))$. Similarly, the average expected discounted payoffs induced by a strategy pair, $\sigma = (\sigma_1, \sigma_2)$ are given by $v(\sigma) = E_\sigma[v(h)]$. At any period t, the continuation strategy profile in the next period, $\sigma|h_{t+1}$ is a function of current actions; hence, $E_{\sigma(h_t)}[v(\sigma|h_{t+1})]$ represents the expected continuation payoffs for the players from the next period onward conditioned on the history h_t and the behavioral strategy in period t.

In a repeated game with perfect information, the Perfect Folk Theorem (Fudenberg and Maskin 1986, 1990, 1991), roughly speaking, implies that any strictly individually rational payoff profile $v \in V^*$ can be supported by some subgame-perfect Nash equilibrium when the discount factors of the players are equal and close to 1. Since there are many slightly different versions of this theorem, to avoid confusion I state below the version that I use in this paper.

Perfect Folk Theorem (Fudenberg and Maskin)

Consider a two-person infinitely-repeated game in which public randomization is not available and only the players' choices of action are observable. For any $v_1' \in V_1^*$ and $v_2' \in V_2^*$, there exists a $\underline{\delta} < 1$ such that for all $\delta \in [\underline{\delta}, 1)$ and for all $v \in \{v : V^* \text{ s.t. } v \geq (v_1', v_2')\}$, there is a subgame-perfect Nash equilibrium of the infinitely-repeated game with discount factor δ in which the discounted average payoffs are v.

Remark: This is essentially Proposition 2 in Fundenberg and Maskin (1991). The only difference is that here I emphasize the fact that for any strictly individually rational payoff profile v, a lower bound of the discount factor δ can be found so that any payoff profiles which weakly dominate v can be supported by some subgame-perfect equilibrium. This is obvious from their proof. The idea is that if there exist "punishment" equilibria which can support a certain equilibrium path,

[12] h_{t,s_1,s_2} can be defined recursively as follows: $h_{2,s_1,s_2}(1) = (s_1(h_1), s_2(h_1))$, and $h_{t,s_1,s_2}(s) = (s_1(h_{s,s_1,s_2}), s_2(h_{s,s_1,s_2})), \forall s < t$.

then those "punishment" equilibria should also be sufficient to support any equilibrium path with weakly higher payoffs.

3.2 Repeated Games with Commitment Types

Suppose there is a small probability that Player 1 is a commitment type γ_1 where, for convenience, γ_1 refers to both the commitment type and the corresponding commitment strategy. The true type of Player 1 is her private information. Let μ_0^1 represent Player 2's prior belief that Player 1 is a commitment type. (Throughout this paper, μ_0^1 is assumed to be strictly less than 1.) A repeated game with one-sided incomplete information is denoted by $\Gamma(A, g, \delta, \gamma_1, \mu_0^1)$.

After observing the past actions of Player 1, Player 2 will update his belief about Player 1's true type. Let $\mu(h_t)$ be Player 2's subjective belief that Player 1 is a commitment type at the beginning of period t after observing a history h_t. Note that $\mu(h_1) = \mu_0^1$ by definition. Given σ_1, μ_0^1, and γ_1, define $\rho_1 \equiv (1 - \mu_0^1)\sigma_1 + \mu_0^1\gamma_1$; ρ_1 is Player 2's expectation of Player 1's strategy.[13]

In this paper, a commitment strategy can be any pure history-dependent repeated-game strategy (i.e. $\gamma_1 \in S_1$). A commitment strategy is called simple if it consists of playing a fixed stage-game action. The commitment payoff of a commitment strategy γ_1 is the minimum payoff the rational type of Player 1 will receive if she plays γ_1 and Player 2 plays a best response to it. If Player 2 is certain that Player 1 is a commitment type, Player 1 will obtain a payoff at least as large as the commitment payoff. Define $B_{2,\delta}(\gamma_1) \equiv \arg\max_{s_2 \in S_2} v_2(\gamma_1, s_2)$; $B_{2,\delta}(\gamma_1)$ is the set of Player 2's best responses to the repeated-game strategy γ_1. Since the set of best responses to a commitment strategy γ_1 is a function of the discount factor δ, the commitment payoff will be a function of δ as well. Formally, the commitment payoff, $\nu_\delta(\gamma_1)$ is defined as follows:

Definition 2 (Commitment Payoff)

$$\nu_\delta(\gamma_1) \equiv \min_{b_2 \in B_{2,\delta}(\gamma_1)} v_1(\gamma_1, b_2)$$

Let $b_2(\gamma_1)$ denote the best response of Player 2 to γ_1 that gives Player 1 her commitment payoff. The commitment payoff of the continuation commitment strategy $\gamma_1|h_t$ after a history h_t, $\nu_\delta(\gamma_1|h_t)$ can be defined analogously. It is clear that Player 1 will only have incentive to imitate a commitment type if the commitment payoff is above her minmax payoff. For simplicity, I shall assume

[13] I will use $\rho_1|h_t$ to denote Player 2's expectation of Player 1's strategy after a history h_t. Since in equilibrium Player 2's belief about Player 1's type is consistent, $\rho_1|h_t = (1 - \mu(h_t))\sigma_1|h_t + \mu(h_t)\gamma_1|h_t$.

that $\nu_\delta(\gamma_1|h_t) > \underline{v}_1 \quad \forall h_t$. In the case of simple commitment types, since the commitment payoff is independent of δ, I will simply denote it as $\nu(\gamma_1)$.[14]

The objective of this paper is to characterize the set of perfect-Bayesain-equilibrium payoffs when the players are patient, and the prior probability that the informed player is commitment type is small. Below, I first state the standard definition of perfect Bayesian equilibrium and then introduce the slightly modified version I use in this paper.

Definition 3 (Perfect Bayesian Equilibrium (A)) $(\sigma_1, \sigma_2, \mu)$ *is a perfect Bayesian equilibrium of an infinitely-repeated game* $\Gamma(A, g, \delta, \gamma_1, \mu_0^1)$ *if the following conditions hold for all* $h_t \in H$*:*

$$\mu(h_t) = \frac{\mu(h_{t-1})\gamma_1(a_1^{t-1}|h_{t-1})}{\rho(a_1^{t-1}|h_{t-1})}$$

whenever the denominator is non-zero,

$$v_1(\sigma_1|h_t, \sigma_2|h_t) \geq v_1(\sigma_1', \sigma_2|h_t) \quad \forall \sigma_1' \in \Sigma_1,$$

$$v_2(\rho_1|h_t, \sigma_2|h_t) \geq v_2(\rho_1|h_t, \sigma_2') \quad \forall \sigma_2' \in \Sigma_2.$$

Remark: The first condition requires that an unilateral deviation by the uninformed player does not affect his belief about the other player's type. Specifically, he will not change his belief and conclude that the informed player is rational after his own deviation.

Throughout this paper, I set $\mu_t(h_t) = 0$ whenever $\rho(a_1^{t-1}|h_{t-1}) = 0$. Note that this restriction is consistent with the notion of perfect Bayesian equilibrium, which puts no restrictions on such beliefs. Under this restriction, when Player 2 observes an action of Player 1 which is not consistent with the strategies of both the rational and commitment types of Player 1, he will believe with certainty that Player 1 is a rational type. As a result, once the rational type of Player 1 has revealed her type, the continuation game becomes a perfect-information repeated game. This allows me to construct a perfect Bayesian equilibrium in a simple way. Instead of specifying the behavioral strategies at all information sets, I only need to specify (1) the behavioral strategies at information sets where Player 2 believes that there is a strictly positive probability that Player 1 is a commitment type, and (2) the continuation payoffs at information sets immediately after Player

[14] $\nu_{1,\delta}(a_1) \equiv \min_{a_2 \in br(a_1)} g_1(a_1, a_2)$ where a_1 is the commitment action.

1 has revealed that she is rational.

$$\begin{aligned}\text{Define } H(\gamma_1) &= \{h_t : \exists s_2 \in S_2 \text{ s.t. } h_t = h_{t,\gamma_1,s_2}\}, \\ \text{and } G(\gamma_1) &= \{h_{t_1}.a : h_{t_1} \in H(\gamma_1) \text{ and } h_{t_1}.a \notin H(\gamma_1)\}.\end{aligned}$$

$H(\gamma_1)$ is the set of finite histories that are consistent with the commitment strategy γ_1, and $G(\gamma_1)$ is the set of histories which are consistent with the commitment strategy in all periods except the last. Let $\sigma^*(h_t) = (\sigma_1^*(h_t), \sigma_2^*(h_t)) : H(\gamma_1) \to \mathcal{A}_1 \times \mathcal{A}_2$ be a "restricted" behavioral strategy profile. Define $\rho_1^* : H(\gamma_1) \to \mathcal{A}_1$ such that $\rho_1^* \equiv (1-\mu)\sigma_1^* + \mu\gamma_1$. Note that ρ_1^*, σ_1^* and σ_2^* are defined only at information sets consistent with the commitment strategy γ_1. Given any $h_t \in H(\gamma_1)$, σ^* and ρ^* induces a probability distribution over $H_s \cap (H(\gamma_1) \cup G(\gamma_1))$ for all $s > t$. For all $h_t \in H(\gamma_1)$, let $P_{\sigma^*,\rho_1^*}(h_s|h_t)$ be the probability of reaching information set h_s, $h_s \in H(\gamma_1) \cup G(\gamma_1)$, for $s > t$ conditional on h_t being reached. Let $P_{\sigma^*}(h_s|h_t)$ be the probability of reaching information set h_s, $h_s \in H(\gamma_1) \cup G(\gamma_1)$, for $s > t$ conditional on h_t being reached and Player 1 being rational. Finally, let $\beta^*(h_t) = (\beta_1^*(h_t), \beta_2^*(h_t)) : G(\gamma_1) \to V^*$ be the continuation payoffs for the players at $h_t \in G(\gamma_1)$. Given σ^*, μ^*, and β^*, $v_1^*(h_t)$ and $v_2^*(h_t)$, the continuation payoffs for Players 1 and 2, respectively, after a history $h_t \in H(\gamma_1)$ are given by

$$\begin{aligned}v_1^*(h_t) &= (1-\delta)\sum_{s=t}^{\infty}\{\sum_{h_s \in H(\gamma_1)} \delta^{s-t} P_{\sigma^*}(h_s|h_t) g_1(\sigma^*(h_s)) + \sum_{h_s \in G(\gamma_1)} \delta^{s-t} P_{\sigma^*}(h_s|h_t)\beta_1^*(h_s)\} \\ v_2^*(h_t) &= (1-\delta)\sum_{s=t}^{\infty}\{\sum_{h_s \in H(\gamma_1)} \delta^{s-t} P_{\sigma^*,\rho_1^*}(h_s|h_t) g_2(\sigma^*(h_s)) + \sum_{h_s \in G(\gamma_1)} \delta^{s-t} P_{\sigma^*,\rho_1^*}(h_s|h_t)\beta_2^*(h_s)\}\end{aligned}$$

Definition 4 (Perfect Bayesian Equilibrium (B)) *σ^*, β^*, and μ are consistent with the notion of perfect Bayesian equilibrium if the following conditions hold for all $h_t \in H(\gamma_1)$:*

$$\mu(h_t) = \frac{\mu(h_{t-1})\gamma_1(a_1^{t-1}|h_{t-1})}{\rho^*(a_1^{t-1}|h_{t-1})},$$

$$v_1(\sigma_1^*|h_t, \sigma_2^*|h_t) \geq (1-\delta)g_1(a_1, \sigma_2^*(h_t)) + \delta E_{(a_1,\sigma_2^*(h_t))}[v_1(\sigma_1^*|h_{t+1}, \sigma_2^*|h_{t+1})] \quad \forall a_1 \in A_1,$$

$$v_2(\rho_1^*|h_t, \sigma_2^*|h_t) \geq (1-\delta)g_2(\sigma_1^*(h_t), a_2) + \delta E_{(\sigma_1^*(h_t),a_2)}[v_2(\rho_1^*|h_{t+1}, \sigma_2^*|h_{t+1})] \quad \forall a_2 \in A_2, \text{ and}$$

there exists a subgame-perfect Nash equilibrium $\tilde{\sigma}_{h_t}$ with equilibrium payoffs equal to $\beta^*(h_t)$, for all $h_t \in H(\gamma_1)$.

Given σ^*, β^*, and μ that are consistent with the notion of perfect Bayesian equilibrium, a perfect Bayesian equilibrium (σ, μ) with the same equilibrium payoff for the rational players can be constructed by setting $\sigma(h_t) = \sigma^*(h_t) \quad \forall h_t \in H(\gamma_1)$, and $\sigma|h_t = \tilde{\sigma}_{h_t} \quad \forall h_t \in G(\gamma_1)$.

4 Non-Existence of Reputation Effects

The example in Section 2 demonstrates that if the prior probability that Player 1 (the informed player) is a commitment type is small, it may take a long time for her to develop a reputation as a commitment type. In this section, I formalize that intuition and show that in repeated games in which one player possesses a small amount of private information, under fairly general conditions, for any $v_1 \in V_1^*$ there exists a perfect Bayesian equilibrium in which the expected equilibrium payoff for the rational type of the Player 1 is equal to v_1. In other words, when the players are sufficiently patient, any payoff for Player 1 that can be supported by a subgame-perfect Nash equilibrium in a perfect-information repeated game can also be supported by some perfect Bayesian equilibrium when Player 1 possesses a small amount of private information. Since I assume that $\underline{v}_1$ is in the closure of V_1^*, the perfect-Bayesian-equilibrium payoff for Player 1 can be arbitrarily close to $\underline{v}_1$. This section is divided into three parts: The first part considers the case of simple commitment strategies; the second considers the general case of history-dependent commitment strategies; and the last briefly discusses other extensions of the basic result.

4.1 Simple Commitment Types

Theorem 1 shows that when the commitment type is simple, reputation effects do not exist if the stage game is neither a strongly-conflicting-interest game nor a strongly-dominant-action game.

Definition 5 (Strongly-Conflicting-Interest Games) *A stage game (A, g) is a strongly-conflicting-interest game if*

5.1. There exists $\tilde{a}_1 \in A_1$ such that $\forall a_2 \in br(\tilde{a}_1)$, $g_1(\tilde{a}_1, a_2) = \bar{v}_1$ and $g_2(\tilde{a}_1, a_2) = \underline{v}_2$,

5.2. $\underline{v}_2 = \max\{v_2 : (\bar{v}_1, v_2) \in V\}$.

Recall that $\bar{v}_1$ is the highest payoff for Player 1 in V that is consistent with Player 2 getting at least his minmax payoff, and $\underline{v}_2$ is the minmax payoff for Player 2. Condition 5.2 implies that the converse also holds: $\underline{v}_2$ is the highest payoff for Player 2 in V that is consistent with Player 1 getting $\bar{v}_1$. Thus, $(\bar{v}_1, \underline{v}_2)$ is on the Pareto frontier. Condition 5.1 implies that there is an action $\tilde{a}_1$ for Player 1 such that if Player 1 chooses $\tilde{a}_1$ and Player 2 chooses a best response to $\tilde{a}_1$, then Player 1 will receive $\bar{v}_1$ and Player 2 will receive $\underline{v}_2$. Note that the notion of strongly-conflicting-interest games is stronger than the notion of conflicting-interest games introduced by Schimdt (1993). The latter only requires that there is an action $\tilde{a}_1$ for Player 1 such that if Player 1 chooses $\tilde{a}_1$ and Player 2 chooses a best response to it, Player 1 will obtain her Stackelberg payoff and Player 2 will obtain his minmax payoff. The Stackelberg payoff is less than or equal to $\bar{v}_1$. Moreover, the notion of conflicting-interest games does not require that $\underline{v}_2$ be the highest payoff which is consistent with $\bar{v}_1$.

Definition 6 (Strongly-Dominant-Action Games) *A stage game (A, g) is a strongly-dominant-action game if*

6.1. $\exists\, \tilde{a}_1 \in A_1$ *such that* $\forall a_2 \in br(\tilde{a}_1),\ g_1(\tilde{a}_1, a_2) = \bar{v}_1,$

6.2. $\forall a_2 \in A_2,\ \forall a_1 \in A_1/\{\tilde{a}_1\},\ g_1(\tilde{a}_1, a_2) > g_1(a_1, a_2).$

Then $\tilde{a}_1$ is called a strongly-dominant action for Player 1

Condition 6.1 is similar to the first part of Condition 5.1 in the definition of strongly-conflicting-interest games, except it does not put any restrictions on the corresponding payoff to Player 2. Condition 6.2 requires that $\tilde{a}_1$ be a strictly-dominant action for Player 1.

The key of the proof is to construct a strategy profile in which (1) the rational type of Player 1 is willing to reveal her type and (2) Player 2 suffers a long-term loss for not screening, (i.e. q is strictly positive). It is instructive to see how the argument fails when the stage game is either a strongly-conflicting-interest or a strongly-dominant-action game. In both cases, the problem appears in the last period of screening. When Player 1 imitates a commitment type who plays a strongly-dominant action, Player 1 will recognize, in the last period of screening, that if she imitates the commitment type for one more period, she will receive her highest payoff from the next period onward. And since the commitment action is strictly dominant, she will not reveal her type in the last period, violating the first condition. In the next section, I show that in this case Player 1 will always receive the commitment payoff in equilibrium. In the case of a strongly-conflicting-interest game, for Player 1 to be willing to reveal her type at the end of the screening phase, her continuation payoff has to be close to her commitment payoff. The assumption of strongly-conflicting interest implies that Player 2 must receive a payoff close to his minmax payoff if he follows his equilibrium strategy. Since the continuation payoff for Player 2 is always weakly higher than his minmax payoff, there is no scope to further "punish" Player 2 for not screening, violating the second condition.

Theorem 4 *In a two-person infinitely-repeated game with one-sided incomplete information and where the players' discount factors are equal,* $\Gamma(A, g, \delta, \gamma_1, \mu_0^1)$, *if the stage game* (A, g) *is neither a strongly-conflicting-interest game nor a strongly-dominant-action game, and* γ_1 *is a simple commitment type, then* $\forall v_1 \in V_1^*$, $\exists\, \underline{\delta}$ *and* $\bar{\mu}_0^1$, *such that* $\forall \delta \geq \underline{\delta}$ *and* $\mu_0^1 \square \bar{\mu}_0^1$, *there is a perfect Bayesian equilibrium in which the average discounted payoff for the rational type of Player 1 is equal to* v_1.

Proof of Theorem 1

Let a_1^c be the commitment action for Player 1, and a_2^c be an action of Player 2 that belongs to $\arg\min_{a_2 \in br(a_1^c)} g_1(\hat{a}_1, a_2)$. Since $\underline{v}_1$ is by definition the minmax payoff for Player 1, there is an action a_2^s for Player 2 such that $g_1(a_1^c, a_2^s) \square \underline{v}_1$. If the rational type of Player 1 imitates the commitment type by choosing a_1^c when Player 2 chooses a_2^s, she will receive a stage-game payoff below or equal to her minmax payoff. Let $a_1^* \in \arg\max_{a_1 \in A_1/\{a_1^c\}} g_1(a_1, a_2^s)$. That is, a_1^* gives Player 1 the highest stage-game payoff against a_2^s among all non-commitment actions for Player 1. In the equilibrium I describe below, Player 1 will mix between a_1^c and a_1^*. The idea of the proof is to show that when μ_0^1 is small enough, it is possible to construct a restricted strategy profile $\sigma^*(h_t)$, $\forall h_t \in H(\gamma_1)$, a set of continuation payoffs $\beta^*(h_t)$, $\forall h_t \in G(\gamma_1)$, and a system of beliefs $\mu(h_t)$, $\forall h_t \in H$ which are consistent with the notion of perfect Bayesian equilibrium for every discount factor δ above a certain threshold. Hence, implicitly, the σ^*, β^*, and μ^* defined below are all functions of δ.

Since by definition $\bar{v}_1$ is the highest payoff for Player 1 in V that is consistent with Player 2 getting at least his minmax payoff, it is obvious that the commitment payoff $\nu(\gamma_1)$ is less than or equal to $\bar{v}_1$. First, I consider the case where $\nu(\gamma_1) = \bar{v}_1$.

Case 1: $\nu(\gamma_1) = \bar{v}_1$

Note that $g_1(a_1^c, a_2^c) = \bar{v}_1$. Thus, if the stage game is not a strongly-conflicting-interest game, then either the second part of Condition 4.1 or Condition 4.2 is violated. In both cases, there exists $v_2' > \underline{v}_2$ such that $(\bar{v}_1, v_2') \in V^*$. Furthermore, there is $\hat{a}_2 \in A_2$ and $\hat{a}_1 \in A_1$ such that (1) $g_1(\hat{a}_1, \hat{a}_2) > g_1(a_1^c, \hat{a}_2)$, and (2) $g_1(a_1^c, \hat{a}_2) < \bar{v}_1$. (1) follows from the fact that the stage game is not a strongly-dominant-action game. To see (2), suppose to the contrary, that $g_1(a_1^c, \hat{a}_2) \geq \bar{v}_1$; then from (1), $g_1(\hat{a}_1, \hat{a}_2) > \bar{v}_1$. This implies that there exists $\lambda \in (0, 1)$ such that $\lambda(\bar{v}_1, v_2') + (1 - \lambda)(g_1(\hat{a}_1, \hat{a}_2), g_2(\hat{a}_1, \hat{a}_2)) \gg (\bar{v}_1, \underline{v}_2)$, which contradicts the definition of $\bar{v}_1$.

Since, by assumption, $\underline{v}_1$ is in the closure of V_1^*, there is $v_2'' \geq \underline{v}_2$ such that $(\underline{v}_1, v_2'') \in V$. Let

$f : V_1 \rightarrow \Re$ be a linear function that represents the line passing through $(\underline{v}_1, v_2'')$ and $(\bar{v}_1, v_2')$. Let $(\check{v}_1, f(\check{v}_1))$ and $(\hat{v}_1, f(\hat{v}_1))$ be two points on the line segment so that $\check{v}_1 \in (\underline{v}_1, v_1)$ and $f(\hat{v}_1) \in (\underline{v}_2, \min\{f(v_1), f(\bar{v}_1)\})$. Set $q = \min\{f(v_1), f(\bar{v}_1)\} - f(\hat{v}_1)$; q is strictly positive. Note that, by construction, $\check{v}_1 \in V_1^*$ and $f(\hat{v}_1) \in V_2^*$; hence, it follows from the Perfect Folk Theorem that any payoff profile, (v_1, v_2), that is strictly bigger than $(\check{v}_1, f(\hat{v}_1))$ can be supported by a subgame-perfect Nash equilibrium when the players are sufficiently patient.[15] First, I show that it is possible to construct a perfect Bayesian equilibrium in which the equilibrium payoff for the rational type of Player 1 is equal to $v_1^1 \in [v_1 - (1-\delta)d, v_1] \subset V_1^*$. See Figure 3.

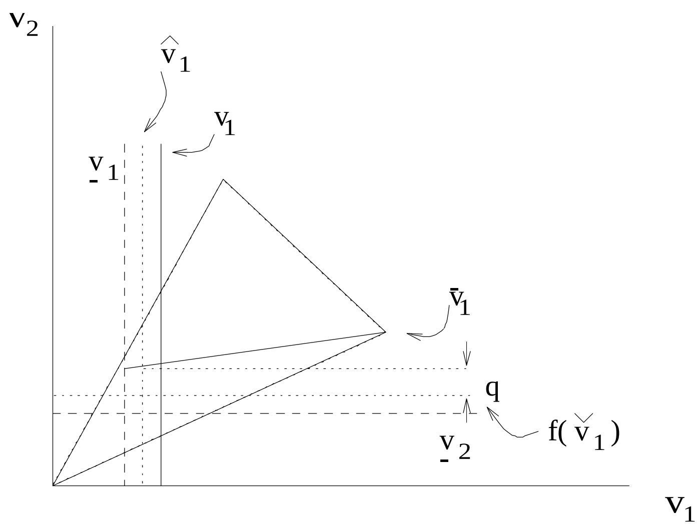

Figure 3: Stage-game Payoff when $\nu(\gamma_1) = \bar{v}_1$

Let δ^* be the smallest number such that $\forall\ \delta \in [\delta^*, 1)$, the following conditions hold: (a) $\bar{v}_1 - \frac{1-\delta}{\delta}(d+1) \geq g_1(a_1^c, \hat{a}_2)$, (b) $\bar{v}_1 - \frac{1-\delta}{\delta}(d+1) - (1-\delta)d \geq v_1$, and (c) $[v_1 - (1-\delta)(d+1), v_1 + (1-\delta)(d+1)] \subset [\check{v}_1, \bar{v}_1]$. These conditions impose a lower bound on δ for the equilibrium strategies I construct below. For the rest of the proof, assume $\delta \geq \delta^*$.

The "restricted" equilibrium strategies of the two players σ_1^*, σ_2^* defined in $H(\gamma_1)$ (as a function of δ) are similar to those described in the example in Section 2. During the first $T_1 + T_2$ periods, Player 2 will choose a non-best-response to a_1^c until Player 1 has revealed that she is rational by choosing an action other than a_1^c. I refer to these periods where Player 2, uncertain about Player 1's type, chooses a non-best-response to commitment strategy as the "screening" phase, and the

[15]This is important because $\underline{v}_1$ and $\underline{v}_2$ may not belong to set of subgame-perfect Nash equilibrium payoffs.

action chosen by Player 2 as a "screening" action. In this case, the screening phase consists of two parts: during the first T_1 periods, Player 2 will play a_2^s , and during the next T_2 periods, he will play $\hat{a}_2$. The two phases serve different functions. During each period of the first phase, the rational type of Player 1 will receive a payoff lower than or equal to her minmax payoff if she imitates the commitment type. By choosing an appropriate T_1, the equilibrium payoff for the rational type of Player 1 can be held down close to her minmax payoff. While a_2^s is an efficient way to "punish" the rational type of Player 1 for imitating the commitment type, it does not provide incentive to induce the rational type of Player 1 to reveal her type. It is possible that the commitment action may already be a best response to a_2^s. Thus, a second screening action $\hat{a}_2$ is needed to induce the rational type of Player 1 to reveal her type near the end of the screening phase.

First, T_2 is set so that the continuation payoff for Player 1 in the beginning of the second screening phase is less than $\bar{v}_1 - \frac{1-\delta}{\delta}d$. This is necessary in order to provide enough incentive for the rational type of Player 1 to reveal her type near the end of the first screening phase. Set $T_2 \equiv \min\{t : (1-\delta^{t-1})g_1(a_1^c, \hat{a}_2) + \delta^{t-1}(1-\delta)g_1(\hat{a}_1, \hat{a}_2) + \delta^t \bar{v}_1 \;\square\; \bar{v}_1 - \frac{1-\delta}{\delta}d\}$. T_2 exists because of (a). Next, T_1 is set so that in equilibrium if the rational type of Player 1 chooses the commitment strategy forever, her average payoff will be (slightly) lower than v_1^1. Define the function - $: \mathcal{Z}_+ \rightarrow \Re$ as follows:

$$- (t) \equiv (1-\delta^t)g_1(a_1^c, a_2^s) + \delta^t\{(1-\delta^{T_2})g_1(a_1^c, \hat{a}_2) + \delta^{T_2-1}(1-\delta)g_1(\hat{a}_1, \hat{a}_2) + \delta^{T_2}\bar{v}_1\} \qquad (8)$$

If the first screening phase lasts for t periods and the second screening phase lasts for T_2 periods, then - (t) is the average payoff the rational type of Player 1, if she mimics the commitment type in the first $t + T_2 - 1$ periods, chooses $\hat{a}_1$ in period $t + T_2$, and obtains her commitment payoff from period $t + T_2 + 1$ onward. Note that - $(0) = (1-\delta^{T_2})g_1(a_1^c, \hat{a}_2) + \delta^{T_2-1}(1-\delta)g_1(\hat{a}_1, \hat{a}_2) + \delta^{T_2}\bar{v}_1 \geq \bar{v}_1 - \frac{1-\delta}{\delta}d - (1-\delta)d \geq v_1$. The first inequality follows from the definition of T_2, and the second inequality follows from (b). Moreover, since - $(t-1)$ – - $(t) = (1-\delta)\delta^t(\bar{v}_1 - g_1(a_1^c, a_2^s)) \;\square\; (1-\delta)d$ and $\lim_{t\to\infty}$ - $(t) \;\square\; \underline{v}_1$, there exists t such that - $(t) \in [v_1 - (1-\delta)d, v_1] \subset V_1^*$. Let T_1 be the smallest such t.

Set $\Delta \equiv \frac{(1-\delta)d}{\delta q}$. As in the example in Section 2, Δ is the probability that the screening action will be chosen in each period during the screening phase. It follows from Lemma 2.3 that there are $\bar{\mu}_0$ and δ^{**} such that $\forall \mu_0 \;\square\; \bar{\mu}_0$ and $\delta \geq \delta^{**}$, $\frac{\mu_0}{(1-\Delta)^{T_1+T_2}} \;\square\; 1$. For all $\mu_0 \;\square\; \bar{\mu}_0$ and $\delta \geq \max\{\delta_1^*, \delta^{**}\}$, define σ_1^*, σ_2^*, and μ are given in Tables 1 and 2.

$\mathbf{h_t \in H(\gamma_1)}$	$\boldsymbol{\sigma_1^*}(\mathbf{h_t})$	$\boldsymbol{\sigma_2^*}(\mathbf{h_t})$
1 □ t □ T_1	$\sigma_1^*(a_1^*\|h_t) = \frac{\Delta}{1-\mu(h_t)}$ $\sigma_1^*(a_1^c\|h_t) = (1 - \frac{\Delta}{1-\mu(h_t)})$	$\sigma_2^*(a_2^s\|h_t) = 1$
$T_1 < t < T_1 + T_2$	$\sigma_1^*(\hat{a}_1\|h_t) = \frac{\Delta}{1-\mu(h_t)}$ $\sigma_1^*(a_1^c\|h_t) = (1 - \frac{\Delta}{1-\mu(h_t)})$	$\sigma_2^*(\hat{a}_2\|h_t) = 1$
$t = T_1 + T_2$	$\sigma_1^*(\hat{a}_1\|h_t) = 1$	$\sigma_2^*(\hat{a}_2\|h_t) = 1$
$t > T_1 + T_2$	$\sigma_1^*(a_1^c\|h_t) = 1$	$\sigma_2^*(a_2^c\|h_t) = 1$

Table 1: The equilibrium strategies, $\sigma^* \quad \forall h_t \in H(\gamma_1)$

$\mathbf{h_t \in H}$	$\boldsymbol{\mu^*}(\mathbf{h_t})$
$h_t \in H(\gamma_1)$, 1 □ t □ $T_1 + T_2$	$\frac{\mu_0^1}{(1-\Delta)^{t-1}}$
$h_t \in H/H(\gamma_1)$	0
$h_t \in H(\gamma_1)$, $t > T_1 + T_2$	1

Table 2: Player 2's beliefs about Player 1's type

During each period of the screening phase, Player 1 will choose a_1^c with a probability of $1 - \frac{\Delta}{1-\mu_t}$ and reveal her type (by choosing a_1^* in the first T_1 periods and $\hat{a}_1$ in the next T_2 periods) with a probability of $\frac{\Delta}{1-\mu_t}$, where μ_t represents Player 2's posterior belief that Player 1 is a commitment type in period t. It is straightforward to verify that μ_t is consistent with Bayes' rule. If the rational type of Player 1 has chosen a_1^c through the first T_1+T_2-1 periods, then in period T_1+T_2 according to the equilibrium strategy Player 1 should choose $\hat{a}_1$ with probability 1. Since Player 2 thinks that only the commitment type will choose a_1^c in that period, if he observes Player 1 choosing a_1^c, he will believe that Player 1 is the commitment type. However, no matter what Player 1 chooses, Player 2 will choose a_2^c from period $T_1 + T_2 + 1$ onward. In other words, Player 1 can obtain the commitment payoff from period T_1+T_2+1 onward by choosing $\hat{a}_1$ during the first T_1+T_2-1 periods.

Define $\Lambda(t) : \{1, .., T_1 + T_2\} \rightarrow \Re$ such that:

$$v_1^1 = (1-\delta)\sum_{i=1}^{t-1}\delta^{i-1}g_1(a_1^c, \sigma_2^*) + \delta^{t-1}\Lambda(t). \tag{9}$$

$\Lambda(t)$ is the equilibrium continuation payoff for the rational type of Player 1 in period t during the screening phase.

If Player 1 reveals that she is rational for the first time in period t, then the continuation game from period $t+1$ onward becomes a standard complete-information repeated game. The continuation equilibrium payoffs are set so that Player 2 has an incentive to screen. Partition $G(\gamma_1)$, the set of information sets immediately after Player 1 has revealed that she is rational, into two subsets: $G_1 \equiv \{h_t.a^t \in G(\gamma_1) : \ a_2^t \neq \sigma_2^*(h_t)\}$, and $G_2 \equiv \{h_t.a^t \in G(\gamma_1) : \ a_2^t = \sigma_2^*(h_t)\}$. The set G_1 contains the periods in $G(\gamma_1)$ where Player 2 has deviated in the last period, and the set G_2 contains the periods in $G(\gamma_1)$ where Player 2 has not deviated in the last period. The continuation payoffs depend on whether Player 2 has deviated from the equilibrium strategy in the last period. For all $H_t \in G_1$, set $\beta^*(h_t) = (\tilde{v}_1, f(\tilde{v}_1))$. Recall that, by definition, $\underline{v}_1 < \tilde{v}_1 < \nu(\gamma_1)$. The continuation payoffs $\beta^*(h_t)$ for all $h_t \in G_2$ are given in Table 3.

$\mathbf{h_t \in G_2}$	$\boldsymbol{\beta_1^*}(\mathbf{h_t})$	$\boldsymbol{\beta_2^*}(\mathbf{h_t})$
$1 \leq t \leq T_1 + 1$	$\Lambda(t) - \frac{1-\delta}{\delta}(g_1(a_1^c, a_2^s) - g_1(a_1^*, a_2^s))$	$f(\beta_1^*(h_t))$
$T_1 + 2 \leq t \leq T_1 + T_2$	$\Lambda(t) - \frac{1-\delta}{\delta}(g_1(a_1^c, \hat{a}_2) - g_1(\hat{a}_1, \hat{a}_2))$	$f(\beta_1^*(h_t))$
$t = T_1 + T_2$	$\bar{v}_1$	$f(\bar{v}_1)$
$t > T_1 + T_2$	$\tilde{v}_1$	$f(\tilde{v}_1)$

Table 3: Continuation Payoffs, $\beta^*(h_t)$, $\quad \forall h_t \in G_2$

The "restricted" equilibrium strategies σ_1^* and σ_2^* are sequentially rational given μ and β^*. First, consider the case of Player 1. In period $T_1 + T_2$, no matter what she chooses, she will get a continuation payoff of $\bar{v}_1$ from period $T_1 + T_2 + 1$ onward. She will get a higher stage-game payoff by revealing her type because $g_1(a_1^c, \hat{a}_2) < g_1(\hat{a}_1, \hat{a}_2)$. Hence, in period $T_1 + T_2$, it is optimal for Player 1 to follow the equilibrium strategy and reveal her type. In any period t between $T_1 + 1$ and $T_1 + T_2 - 1$, if Player 1 chooses a_1^c and follows the equilibrium strategy after that, her payoff will be equal to $(1-\delta)g_1(a_1^c, \hat{a}_2) + \delta\Lambda(t)$; if Player 1 chooses $\hat{a}_1$ and follows the equilibrium strategy after that, her payoff will be equal to $(1-\delta)g_1(\hat{a}_1, \hat{a}_2) + \delta\beta_1^*(h_{t-1}.(\hat{a}_1, \hat{a}_2))$. Similarly, in any period t between 1 and T_1, if Player 1 chooses a_1^c and follows her equilibrium strategy after that, her payoff will be equal to $(1-\delta)g_1(a_1^c, a_2^s) + \delta\Lambda(t)$; if Player 1 chooses a_1^* and follows her equilibrium strategy after that, her payoff will be equal to $(1-\delta)g_1(a_1^*, a_2^s) + \delta\beta_1^*(h_{t-1}.(a_1^*, a_2^s))$. It is easy to verify that $\beta_1^*(h_t)$ is defined so that Player 1 is indifferent between a_1^c and a_1^* in the first T_1 periods, and between a_1^c and $\hat{a}_1$ in the next $T_2 - 1$ periods. Next, consider the case of Player 2. By construction, his action in any period t during the screening phase will only affect his stage-game payoff and the continuation payoff in the event that Player 1 reveals that she is rational in the same period. Hence, the short-term cost of screening is less than $(1-\delta)d$, while the long-term gain from screening is larger then $\delta\Delta q$. It is easy to verify that σ_2^* is optimal for Player 2.

Lastly, to complete the proof, I need to show that $\beta^*(h_t)$ can be supported by some subgame-perfect Nash equilibrium for all $h_t \in G_1 \cup H_2$. For all $t \,\square\, T_1$, From (1) and (2) and the definition of T_2, it follows that

$$\Lambda(t) \,\square\, (1-\delta^{t-1})g_1(a_1^c, \hat{a}_2) + \delta^{t-1}(1-\delta)g_1(\hat{a}_1, \hat{a}_2) + \delta^t\bar{v}_1 \,\square\, \bar{v}_1 - \frac{1-\delta}{\delta}d$$
$$\Rightarrow \beta_1^*(h_t) \,\square\, \bar{v}_1 - \frac{1-\delta}{\delta}d + \frac{1-\delta}{\delta}(g_1(a_1^c, a_2^s) - g_1(a_1^*, a_2^s)) \,\square\, \bar{v}_1.$$

By definition $\Lambda(t) \,\square\, \bar{v}_1$ for all $T_1 + 1 \,\square\, t \,\square\, T_1 + T_2$. In addition, since $g_1(a_1^c, \hat{a}_2) - g_1(\hat{a}_1, \hat{a}_2)) < 0$; then,

$$\beta_1^*(t) \,\square\, \bar{v}_1 \quad \forall\, T_1 + 1 \,\square\, t \,\square\, T_1 + T_2$$

As a result, $\beta^*(h_t) \in V^*$ for all $h_t \in G_2$. Moreover, since $\beta^*(h_t) >> (\hat{v}_1, f(\bar{v}_1))$, it follows from the Perfect Folk Theorem that there exists δ^{***} so that for all $\delta \geq \delta^{***}$, any $\beta^*(h_t) \quad h_t \in G_1 \cup G_2$ can be supported by some subgame-perfect Nash equilibrium.

In summary, I have shown that for all $\delta \geq \max\{\delta^*, \delta^{**}, \delta^{***}\}$ and $\mu_0 \,\square\, \bar{\mu}_0$, I can construct a perfect Bayesian equilibrium with the equilibrium payoff for the rational type of Player 1 equal to $v_1^1 \in [v_1 - (1-\delta)d, v_1]$. Using the same argument, I can construct another equilibrium in which the first period action of Player 2 is different from the equilibrium above, and the equilibrium payoff for the rational type of Player 1 equal to $v_1^2 \in [v_1, v_1 + (1-\delta)d] \in V_1^*$. Finally, an equilibrium with the payoff for the rational Player 1 exactly equal to v_1 can be constructed by taking a convex combination of the two equilibria. This can be achieved by making Player 2 mix with an appropriate probability between the two equilibria in the first period.

Case 2 $\nu(\gamma_1) < \bar{v}_1$

The construction of the desired equilibrium is simpler in this case. When there is a gap between the commitment payoff and $\bar{v}_1$, the continuation payoffs for the rational type of Players 1 at the end of the "screening" phase can be adjusted to induce her to reveal her type, and give her the desired equilibrium payoff . Thus, it would be unnecessary to construct two different equilibria as in Case 1. Since Case 2 is a special case of Theorem 2 in the next section, I shall not repeat the proof here. For details, see the proof of Theorem 2. □

4.2 History-Dependent Commitment Strategies

In many games, a player may want to commit to a history-dependent strategy. For example, in the infinitely-repeated Prisoners' Dilemma, while a player obviously would not wish to commit to simple strategies of either cooperating or defecting unconditionally, she may like to commit to a history-dependent "Tit for Tat" strategy.

When the commitment strategy of Player 1 is history-dependent, Player 2's action in one period may affect the commitment type's continuation strategy and, hence, the long-term payoff for Player 2. As a result, the proof of Theorem 1 does not apply directly. However, I show that the conclusion of Theorem 1 continues to hold under a stronger sufficient condition when the commitment strategy is history-dependent.

In order to understand the complication that arises when the commitment strategy is history-dependent, we need to get into the details of the basic structure of the equilibrium I construct in the proof of Theorem 1. The decision tree in Fig. 4 represents the decision facing Player 2 in period t during the screening phase. In each period, Player 2 expects both types of Player 1 to

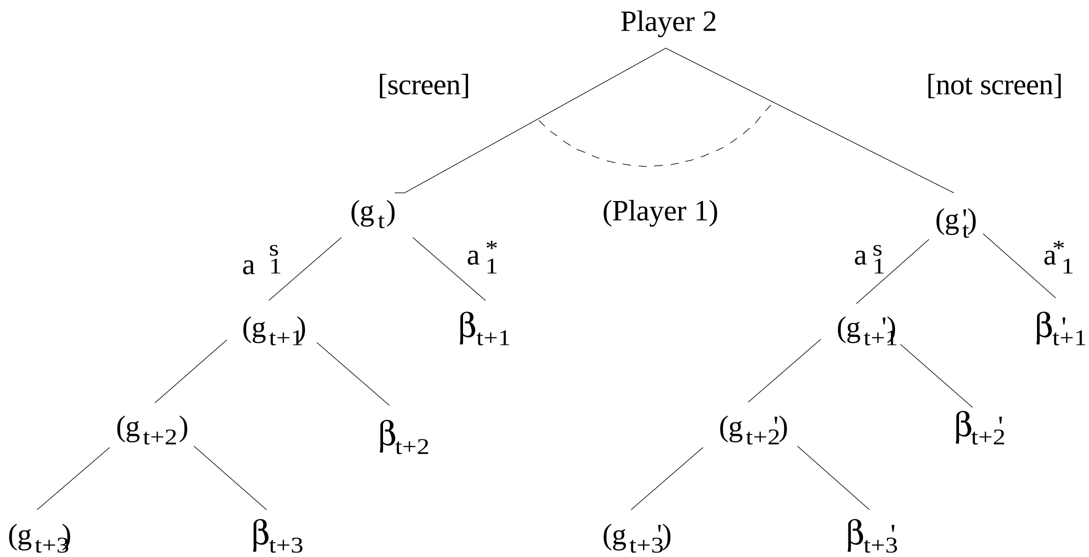

Figure 4: Player 2's decision in period t

choose the commitment action with a total probability of $1-\Delta$ and a non-commitment action with a total probability of Δ. If Player 2 chooses a_1^s, he will receive a payoff of g_t in that period. The equilibrium continuation payoff depends on the actions of Player 1. Let g_{t+i} be Player 2's expected equilibrium stage-game payoff in period $t+i$ if Player 1 has not revealed that she is rational before that period, and β_{t+i} be the equilibrium continuation payoff from period $t+i$ onward if Player 1 has revealed that she is rational in period $t+i-1$. The expected continuation equilibrium payoff v_t takes the following form:

$$\begin{aligned} v_t &= \delta[(1-\delta)(1-\Delta)g_{t+1} + \Delta\beta_{t+1}] + \delta^2(1-\Delta)[(1-\delta)(1-\Delta)g_{t+2} + \Delta\beta_{t+2}] \\ &\quad + \delta^3(1-\Delta)^2[(1-\delta)(1-\Delta)g_{t+3} + \Delta\beta_{t+3}] + ... \end{aligned}$$

Similarly, let g'_t, v'_t be the stage-game payoff in period t and the continuation equilibrium payoff for Player 2 if he deviates and chooses a best response to the commitment strategy in period t, but follows the equilibrium strategy from period $t+1$ onward. The continuation payoff v'_t can also be represented in the following form:

$$\begin{aligned} v'_t &= \delta[(1-\delta)(1-\Delta)g'_{t+1} + \Delta\beta'_{t+1}] + \delta^2(1-\Delta)[(1-\delta)(1-\Delta)g'_{t+2} + \Delta\beta'_{t+2}] \\ &\quad + \delta^3(1-\Delta)^2[(1-\delta)(1-\Delta)g'_{t+3} + \Delta\beta'_{t+3}] + ... \end{aligned}$$

When the commitment strategy is simple, Player 2's action in period t will not affect the future actions of the commitment type. Moreover, in the proof of Theorem 1, since the continuation strategies of the rational type of Player 1 and Player 2 are constructed so that they are independent of the previous actions of Player 2, Player 2's future equilibrium stage-game payoffs in the periods where Player 1 has not been revealed as rational are also independent of Player 2's previous actions, i.e. $g_{s+1} = g'_{s+1}$ for $s \geq t$. Hence, I can keep the continuation equilibrium payoffs v_t and v'_t equal by setting $\beta_{s+2} = \beta'_{s+2}$ for all $s \geq t$. As a result, Player 2 will be "punished" for not screening in period t only in the event that Player 1 reveals that she is rational in period t. As I have showed in the example in Section 2, it is rational for Player 2 to screen if

$$(1-\delta)(g'_t - g_t) \,\square\, \Delta\delta(\beta_{t+1} - \beta'_{t+1}) \tag{10}$$

This relation puts a lower bound on Δ, and hence, on the rate of Bayesian updating. In order to ensure that screening can last for many periods, it is crucial that the cost of one period of screening is on the order of $(1-\delta)d$. When the commitment strategy is history dependent, since Player 2's action in period t can affect the future actions of the commitment type, in general, g_{s+1} will not be equal to g'_{s+1} for $s \geq t$. As a result, one period of screening may involve a long-term cost. One way to avoid this problem is to keep the continuation payoffs v_1 and v'_1 equal by setting β_{s+i} and β'_{s+i} for all $s \geq t$ so that

$$[(1-\delta)(1-\Delta)g_{s+i} + \Delta\beta_{s+i}] = [(1-\delta)(1-\Delta)g'_{s+i} + \Delta\beta'_{s+i}]$$

In this case, β_{s+i} will not be equal to β'_{s+i}. Intuitively, Player 2's "punishment" for a deviation is distributed according to how the deviation affects the future actions of the commitment type. Suppose the continuation payoff for the rational type of Player 1 is given by x. In equilibrium, x has to be set so that she is indifferent between mimicking the commitment type and revealing her type and, therefore, cannot be adjusted freely. In order to be able set β_{s+i} and β'_{s+i} according to the equation above, both (x, β_{s+i}) and (x, β'_{s+i}) have to belong to V^*. This means that x cannot equal to $\bar{v}_1$. See Figure 5. The following assumption guarantees that the commitment payoff is strictly less than $\bar{v}_1$.

Assumption 1

$$\bar{\nu}(\gamma_1) < \bar{v}_1$$

where $\bar{\nu}(\gamma_1) \equiv \sup\{\nu_\delta(\gamma_1|h_t)\ \forall h_t \in H, \delta \in (0,1)\}$. When the commitment strategy is history dependent, the continuation commitment payoff after period t depends on the history h_t and the discount factor δ. And $\bar{\nu}(\gamma_1)$ is the supremum of the set of continuation commitment payoffs (including the original commitment payoff at the beginning of the game) after any finite history and for any discount factor.

I show that in any two-person infinitely-repeated game with one-sided incomplete information, if the commitment strategy is pure (but possibly history-dependent) and satisfies Assumption 1, any then $v_1 \in V_1^*$ can be supported by some perfect Bayesian equilibrium when the players are sufficiently patient and the prior probability that Player 1 is a commitment type is sufficiently small. For example, in the *infinitely-repeated* Prisoners' Dilemma, my result implies that Player 1 cannot guarantee herself a payoff close to the "cooperation" payoff by mimicking a "Tit for Tat" type (or any other commitment types with the same commitment payoff).[16]

Note that Assumption 1 is a fairly strong assumption. In strongly-conflicting-interest games and strongly-dominant-actions games, it rules out simple commitment types with commitment payoffs equal to $\bar{v}_1$. More generally, in games where $\bar{v}_1 \in V_1^*$ (e.g. Battle of Sexes), Assumption 1 directly rules out commitment types with commitment payoff equal to $\bar{v}_1$. In games where $\bar{v}_1 \notin V_1^*$ (e.g. Prisoners' Dilemma), where the commitment payoff is, by definition, strictly lower than $\bar{v}_1$, Assumption 1 requires that the commitment payoff be bounded away from $\bar{v}_1$ as δ approaches one.

Theorem 5 *In a two-person infinitely-repeated game with incomplete information* $\Gamma(A, g, \delta, \gamma_1, \mu_0)$ *if* γ_1 *is a pure history-dependent strategy with* $\bar{\nu}(\gamma_1) < \bar{v}_1$*, then* $\forall v_1 \in V_1^*$*,* $\exists\ \underline{\delta}$ *and* $\bar{\mu}_0$*, such that* $\forall \delta \geq \underline{\delta}$ *and* $\mu_0 \square \bar{\mu}_0$*, there is a perfect Bayesian equilibrium in which the average discounted payoff for the rational type of Player 1 is equal to* v_1*.*

Proof of Theorem 2:

Since $\bar{\nu}(\gamma_1) < \bar{v}_1$ and $v_1 > \underline{v}_1$, there exists $\hat{v}_1$, $\overset{\square}{v}_1$ and $\epsilon > 0$ such that $\underline{v}_1 < \overset{\square}{v}_1 \square \bar{v}_1 - \epsilon$ and $\bar{\nu}(\gamma_1) + \epsilon \square \hat{v}_1 < \bar{v}_1$. Since V^* is convex and non-empty, both $\overset{\square}{v}_1$, $\hat{v}_1$ belong to V_1^*, i.e. there are $\overset{\square}{v}_2$, $\hat{v}_2$ that are strictly bigger $\underline{v}_2$ so that $(\overset{\square}{v}_1, \overset{\square}{v}_2)$ and $(\hat{v}_1, \hat{v}_2)$ belong to $int(V^*)$. Let $f : V_1 \to \Re$ be the linear function representing the line passing through $(\overset{\square}{v}_1, \overset{\square}{v}_2)$ and $(\hat{v}_1, \hat{v}_1)$. Since V^* is convex, there exists $q > 0$ such that the set $B \equiv \{(v_1, v_2) : v_1 \in [\overset{\square}{v}_1, \hat{v}_1], v_2 \in [f(v_1) - q, f(v_1) + q]\} \subset V^*$. In other words, any point (v_1, v_2) where $v_1 \in [\overset{\square}{v}_1, \hat{v}_1]$, and where the vertical distance from $(v_1, f(v_1))$

[16] Kreps, et al.(1982) show in *finitely-repeated* prisoners' dilemma, when there is some small probability that one of the players is a "Tit for Tat" type, then in all sequential equilibria both players will cooperate until almost the end of the game.

is less than q is strictly individually rational. See Figure 5.

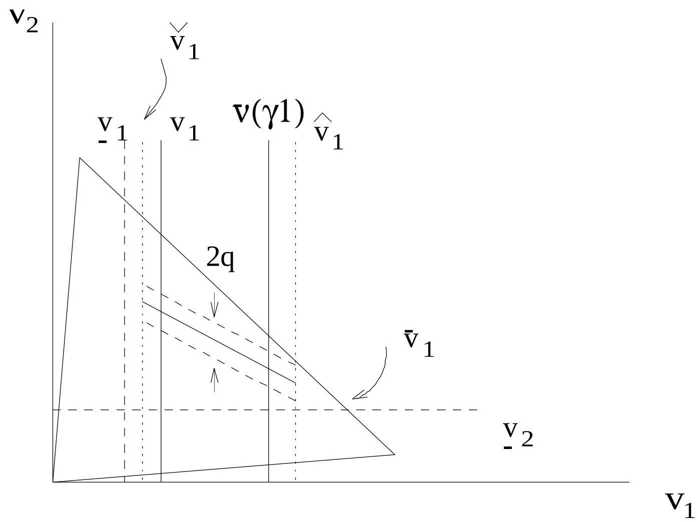

Figure 5: Stage-game Payoff for when $\bar{\nu}(\gamma_1) < \bar{v}_1$

Given γ_1, I can construct a function $r : A_1 \to A_2$ so that $g_1(a_1, r(a_1)) \Box \underline{v}_1$. Define $s_2^* : H(\gamma_1) \to A_2$ such that $s_2^*(h_t) = r(\gamma_1(h_t)) \quad \forall h_t \in H(\gamma_1)$. In any period after some history h_t, if Player 2 chooses $s_2^*(h_t)$ and the rational type of Player 1 chooses $\gamma_1(h_t)$, then the rational type of Player 1 will receive a payoff less than or equal to her minmax payoff.

Below, I assume that the players are patient enough such that the stage-game payoff in any single period does not significantly affect the payoff of the whole repeated game. Specifically, assume that $\frac{1-\delta}{\delta}$ is less than $\frac{\epsilon}{2d}$ and $\frac{q}{2d}$. For any $\delta \geq \delta^* = \min\{\frac{2d}{2d+\epsilon}, \frac{2d}{2d+q}\}$, define $\Lambda : H(\gamma_1) \to \Re$ such that:

$$\Lambda(h_t) = \begin{cases} v_1 & h_t = h_1 \\ \frac{1}{\delta^{t-1}}\{v_1 - (1-\delta)\sum_{i=2}^{t} \delta_1^{i-2} g_1(h_t^1(i-1), r(h_t^1(i-1)))\} & h_t \in H(\gamma_1),\ t \geq 2 \end{cases}$$

In the equilibrium which I shall construct, $\Lambda(h_t)$ will be the continuation payoff for the rational type of Player 1 after a history h_t, if the rational type of Player 1 and Player 2 have played according to γ_1 and s_2^* during the first $t-1$ periods. $\Lambda(h_t)$ is set so that the expected average payoff of the whole game for the rational type of Player 1 is equal to v_1.

Define the following subsets of finite histories:

$$H_1 = \{h_t \in H(\gamma_1) : \ \Lambda(h_t) < \nu(\gamma_1)\}$$

$$H_2 \quad = \quad \{h_t.a \in H(\gamma_1): \; h_t \in H_1 \text{ and } \nu(\gamma_1) \,\square\, \Lambda(h_t) \,\square\, \hat{v}_1 - \frac{\epsilon}{2}\}$$

Screening takes place in periods $h_t \in H_1 \cup H_2$. Consider any complete history of the game h that is consistent with the commitment strategy. If h_{t+1}, h_t are the $t+1-$ and $t-$period truncations of h, then

$$\Lambda(h_t) + \frac{1-\delta}{\delta}(v_1 - \underline{v}_1) \,\square\, \Lambda(h_{t+1}) \,\square\, \Lambda(h_t) + \frac{1-\delta}{\delta} d. \tag{11}$$

The first inequality implies that $\Lambda(h_t)$ is strictly increasing in t, and the second inequality implies that as $\Lambda(h_t)$ increases with t, it will not jump over the interval $[\overline{\nu}(\gamma_1), \overline{v}_1 - \frac{\epsilon}{2}]$, so that there is always a period t^* such that $h_{t^*} \in H_2$. Moreover, for any history that is consistent with the commitment strategy, the number of periods it takes to reach $H_2(\gamma_1)$ is uniformly bounded from above . To see that, note that for all $h_t \in H_1 \cup H_2$, $v_1 \,\square\, (1-\delta^t)\underline{v}_1 + \delta^t \hat{v}_1$; hence, $\delta^t \geq \frac{v_1 - \underline{v}_1}{\hat{v}_1 - \underline{v}_1}$, or equivalently,

$$t \,\square\, \ln\left(\frac{v_1 - \underline{v}_1}{\hat{v}_1 - \underline{v}_1}\right) / \ln \delta.$$

Let T be the smallest integer that is larger than $\ln\left(\frac{v_1 - \underline{v}_1}{\hat{v}_1 - \underline{v}_1}\right) / \ln \delta$. Note that T is implicitly a function of δ. From Lemma 2.3, there exists $\bar{\mu}_0, \delta^{**}$ such that $\forall \mu_0 \,\square\, \bar{\mu}_0^1$, $\delta \geq \delta^{**}$, and $t \,\square\, T(\delta)$,

$$\frac{\mu_0}{(1-\Delta)^t} \,\square\, \bar{\mu} = \frac{q}{2(d+q)}.$$

Set $\Delta \equiv \frac{(1-\delta)}{\delta}\frac{d}{q}$. Now, for all $\mu_0 \,\square\, \bar{\mu}_0$, and $\delta \geq \max\{\delta^*, \delta^{**}\}$, define σ_1^* and σ_2^* in $H(\gamma_1)$, and μ in H according to Tables 4 and 5.

$\mathbf{h_t}$	$\boldsymbol{\sigma_1^*}(\mathbf{h_t})$	$\boldsymbol{\sigma_2^*}(\mathbf{h_t})$
H_1	$\sigma_1^*(a_1^*\|h_t) = \frac{\Delta}{1-\mu(h_t)}$ $\sigma_1^*(\gamma_1(h_t)\|h_t) = (1 - \frac{\Delta}{1-\mu(h_t)})$	$\sigma_2^*(s_2^*(h_t)\|h_t) = 1$
H_2	$\sigma_1^*(a_1^*\|h_t) = 1$	$\sigma_2^*(s_2^*(h_t)\|h_t) = 1$
$H(\gamma_1)/(H_1 \cup H_2)$	$\sigma_1^*(\gamma_1(h_t)\|h_t) = 1$	$\sigma_2^*(b_2(\gamma_1\|h_t)\|h_t) = 1$

Table 4: The equilibrium strategies, $\sigma^*(h_t) \quad \forall h_t \in H(\gamma_1)$

$\mathbf{h_t}$	$\boldsymbol{\mu}(\mathbf{h_t})$
$H_1 \cup H_2$	$\frac{\mu_0^1}{(1-\Delta)^{t-1}}$
$H/H(\gamma_1)$	0
$H(\gamma_1)/H_1 \cup H_2$	1

Table 5: Player 2's beliefs about Player 1's type

In equilibrium, screening takes place in periods with $h_t \in H_1 \cup H_2$. During those periods, Player 2 chooses according to s_2^* so that if the rational type of Player 1 imitates the commitment type, she will at most receives her minmax payoff $\underline{v}_1$.[17] In periods with $h_t \in H_1$, the rational type of Player 1 will choose $\gamma(h_t)$ with a probability of $1 - \frac{\Delta}{1-\mu(h_t)}$ and a_1^* with a probability of $\frac{\Delta}{1-\mu(h_t)}$ Screening ends in periods with $h_t \in H_2$ when the rational type of Player 1 reveals her type by choosing a_1^*. It is easy to verify that μ is consistent with Bayes' rule given σ_1^* and σ_2^*.

The rational type of Player 1 may reveal her type for the first time in one of the following three situations: during the screening phase in H_1, in a final period of screening in H_2, or after the screening phase in $H(\gamma_1)/(H_1 \cup H_2)$. The following subsets of histories correspond to these situations.

$$\begin{aligned} G_3 &= \{h_t.a \in G(\gamma_1) : h_t \in H_1\} \\ G_4 &= \{h_t.a \in G(\gamma_1) : h_t \in H_2\} \\ G_5 &= \{h_t.a \in G(\gamma_1) : h_t \in H(\gamma_1)/(H_1 \cup H_2)\} \end{aligned}$$

The continuation payoffs $\beta^*(h_t) \quad \forall h_t \in G_3 \cup G_4 \cup G_5$ are given in Tables 6 and 7.

It is straightforward to verify that σ_1^* is rational. In periods after the screening phase with $h_t \in H(\gamma_1)/H_1 \cup H_2$, Player 2 is convinced that Player 1 is a commitment type. Since by construction, the continuation commitment payoff is higher than $(1-\delta)d + \delta\overline{v}_1$, the highest payoff the rational type of Player 1 can obtain by deviating, it is optimal for her to follow the equilibrium strategy. During the "screening" phase, by construction, $\Lambda(h_t) = (1-\delta)g_1(a_1^*, s_2^*(h_t)) + \delta\beta_1^*(h'_{t+1})$ for all $h_t \in H_1 \cup H_2$, where $h'_{t+1} = h_t.(a_1^*, s_2^*(h_t))$. That is, if the rational type of Player 1 reveals

[17] She may receive strictly less than $\underline{v}_1$, if she chooses a dominated action.

her type in period t during the "screening" phase, she will receive a payoff of $\Lambda(h_t)$. In periods with $h_t \in H_2$, since $\Lambda(h_t) > \bar{\nu}(\gamma_1)$, Player 1 will strictly prefer to choose a_1^*. In all other periods with $h_t \in H_1$, since $\Lambda(h_t) = (1-\delta)g_1(\gamma_1(h_t), s_2^*(h_t)) + \delta\Lambda(h_{t+1}^*)$ where $h_{t+1}^* = h_t.(\gamma_1(h_t), s_2^*(h_t))$, Player 1 is indifferent between choosing $\gamma(h_t)$ and a_1^*.[18]

$\mathbf{h_t}$	$\boldsymbol{\beta_1^*}(\mathbf{h_t})$	$\boldsymbol{\beta_2^*}(\mathbf{h_t})$
$h_t = h_{t-1}.a \in G_3 \cup G_4$	$\frac{\Lambda(h_{t-1})-(1-\delta)g_1(a_1^*, s_2^*(h_{t-1}))}{\delta}$	$f(\beta_1^*(h_t)) + \eta(h_t)$
$h_t = h_{t-1}.a \in G_5$	$\underline{v}_1$	$f(\underline{v}_1)$

Table 6: Continuation Payoffs $\beta^*(h_t) \quad \forall h_t \in G_3 \cup G_4 \cup G_5$

$\mathbf{h_t}$	$\boldsymbol{\eta}(\mathbf{h_t})$
$h_t = h_{t-1}.a \in G_3$	$\frac{q}{d}\{f(g_1(\rho_1^*(h_{t-1}), s_2^*(h_{t-1}))) - g_2(\rho_1^*(h_{t-1}), a_2)\}$
$h_t = h_{t-1}.a \in G_4$	$\frac{\mu(h_{t-1})}{1-\mu(h_{t-1})}\left\{\frac{f(\Lambda(h_{t-1}))-(1-\delta)f(g_1(\gamma_1(h_{t_1}), s_2^*(h_{t-1}))}{\delta} - v_2(\gamma_1 \vert h_t, b_2(\gamma_1 \vert h_t))\right\} + \frac{1}{1-\mu(h_{t-1})}\frac{1-\delta}{\delta}\{f(g_1(\rho_1^*(h_{t-1}), s_2^*(h_{t-1}))) - g_2(\rho^*(h_{t-1}), a_2)\}$

Table 7: $\eta(h_t) \quad \forall h_t \in G_3 \cup G_4$

Now I shall show that σ_2^* is rational. Let $v_2^*(h_t)$ denote $v_2(\rho_1^*|h_t, s_2^*|h_t)$, the equilibrium continuation payoff in a period with history $h_t \in H_1 \cup H_2$. First, consider any period with $h_t \in H_2$. If $h_t \in H_2$, then by definition, $h_t.(a_1^*, a_2)$ must belong to G_4. If Player 2 chooses a_2 in that period and follows the equilibrium strategy afterward, his payoff will be equal to

$$(1-\delta)g_2(\rho_1^*(h_t), a_2) + \delta\{\mu(h_t)v_2(\gamma_1|h_{t+1}, b_2|h_{t+1}) + (1-\mu(h_t))(f(\beta_2^*(h'_{t+1})) + \eta(h'_{t+1}))\} \quad (12)$$

where $h_{t+1} = h_t.(\gamma_1(h_t), a_2)$ and $h'_{t+1} = h_t.(a_1^*, a_2)$.

By substituting the definition of $\beta^*(h'_{t+1})$ and $\eta(h'_{t+1})$ into the equation and rearranging terms, it

[18] Recall that a_1^* is by definition the best response to the screening action among the non-commitment actions. Since the continuation payoff for the rational type of Player 1 after she has revealed her type does not depend on which non-commitment action she chose, Player 1 strictly prefers a_1^* to any other non-commitment action.

can shown that Equation (12) is equivalent to

$$
\begin{aligned}
&f(\Lambda(h_t)) - \delta\mu(h_t)\{(v_2(\gamma_1|h^*_{t+1}, b_2(\gamma_1|h^*_{t+1})) - v_2(\gamma_1|h_{t+1}, b_2(\gamma_1|h_{t+1}))\} \\
&+(1-\delta)\{f(g_1(\rho^*_1(h_t), s^*_2(h_t)) - g_2(\rho^*(h_t), a_2))\} + (1-\mu(h_t))\delta\eta(h'_{t+1}) \\
&= f(\Lambda(h_t))
\end{aligned}
$$

where $h^*_{t+1} = h_t.(\gamma_1(h_t), s^*_2(h_t))$ and $h'_{t+1} = h_t.(a^*_1, s^*_2(h_t))$. Player 2's stage-game action affects his expected stage-game payoff in that period and the continuation payoff in the next period in the event that Player 1 is a commitment type. The value of $\eta(h'_{t+1})$ $h'_{t+1} \in H_4$ is set to cancel out these effects so that Player 2 is indifferent between choosing any stage-game actions, and his equilibrium continuation payoff is equal to $\Lambda(h_t)$.

Next, I shall show that in any period t with $h_t \in H_1$,

$$
(1-\delta)g_2(\rho^*_1(h_t), a_2) + \delta\{\Delta\beta^*_2(h_{t+1}) + (1-\Delta)v^*_2(h_{t+t})\} = f(\Lambda(h_t)) \quad \forall a_2 \in A_2 \tag{13}
$$

where $h_{t+1} = h_t.(\gamma_1(h_t), a_2)$ and $h'_{t+1} = h_t.(a^*_1, a_2)$. Equation (13) implies that Player 2 is indifferent between any actions $a_2 \in A_2$, and the equilibrium continuation payoff for Player 2, $v^*_2(h_t)$, is equal to $f(\Lambda(h_t))$. Construct $\{K_i\}^T_1$, a finite sequence of subsets of H_1 as follows:

$$
K_1 = \{h_t \in H_1 : \ \forall a_2 \in A_2, \ h_t.(\gamma_1(h_t), a_2) \in H_2\},
$$

and in general,

$$
K_i = \{h_t \in H_1 : \ \forall a_2 \in A_2, \quad h_t.(\gamma_1(h_t), a_2) \in K_{i-1}\} \cup K_{i-1} \ 2 \,\square\, i \,\square\, T.
$$

K_i includes all periods from which it takes Player 1 at most i more periods of mimicking to reach the final period of screening. Since screening lasts for at most T periods, $K_T = H_1$.

Suppose Equation (13) holds, and hence, $v^*_2(h_t) = f(\Lambda(h_t))$ for all $h_t \in K_i$. Consider a period $h_t \in K_{i+1}/K_i$. If Player 2 chooses a_2 in that period, and follows the equilibrium strategy after that, his expected payoff will be equal to

$$
\begin{aligned}
&(1-\delta)g_2(\rho^*_1(h_t), a_2) + \delta\{\Delta\beta^*_2(h_t.(a^*_1, a_2))] + (1-\Delta)f(\Lambda(h_t.(\gamma_1(h_t), a_2))\} \\
&= f(\Lambda(h_t)) + (1-\delta)(g_2(\rho^*_1(h_{t-1}), a_2) - f(g_1(\rho^*_1(h_{t-1}), s^*_2(h_{t-1})))) + \delta\Delta\eta(h_t.(a^*_1, a_2))\} \\
&= f(\Lambda(h_t)).
\end{aligned}
$$

Since I have already shown that $v_2^*(h_t) = f(\Lambda(h_t)) \quad \forall h_t \in H_2$, it is straightforward to verify that Equation (13) holds for all $h_t \in K_1$; therefore, by induction, it holds for all $h_t \in H_1$.

To complete the proof, I need to show that any $\beta^*(h_t)$, $h_t \in G_3 \cup G_4 \cup G_5$, can be supported by some subgame-perfect Nash equilibrium. Since $v_1 = \Lambda(h_1) < \Lambda(h_t) \square \hat{v}_1 - \frac{\epsilon}{2} \quad \forall h_t \in H_1 \cup H_2$, it follows from the definition of $\underline{v}_1$ and $\hat{v}_1$ that

$$\underline{v}_1 \square \beta_1(h_t) \square \hat{v}_1 \quad h_t \in G_3 \cup G_4$$

By construction, if $v_1 \in [\underline{v}_1, \hat{v}_1]$, and $v_2 \in [f(v_1) - q, f(v_1) + q]$, then $(v_1, v_2) \in V^*$. Since by assumption, $f(g_1(\rho_1^*(h_{t-1}), s_2^*(h_{t-1}))) - g_2(\rho_1^*(h_{t-1}), a_2) \square d$, so $|\eta(h_t)| \square q \quad \forall h_t \in G_3$. Similarly, for all $h_t \in G_4$, note that, $\mu(h_t) \square \frac{q}{2(d+q)}$, and $\delta \geq \frac{2d}{2d+q}$,

$$|\eta(h_t)| < \frac{\mu(h_{t_1})}{1 - \mu(h_{t-1})} d + \frac{1}{1 - \mu(h_{t-1})} \frac{1 - \delta}{\delta} d \square q.$$

I have shown that $\beta^*(h_t) \in V^*$ for all $h_t \in G_3 \cup G_4 \cup G_5$. Thus, the Perfect Folk Theorem implies that there exists a δ^{***}, such that $\forall \delta \geq \delta^{***}$, $\beta^*(h_t) \in V^*$ can be supported. $\square$

4.3 Concluding Remarks

Although Theorems 1 and 2 formally assume that the players are equally patient and there is only one pure commitment type, these assumptions are not crucial to my results. Below, I briefly discuss how my results extend to more general cases.

- To extend my results when there are multiple commitment types, I need to construct an equilibrium in which Player 1 mimics all commitment types with strictly positive probability during the screening phase. Note that Player 2 can distinguish between two possible pure commitment types once they have chosen different actions at some information sets. I only need to make sure that at those information sets, the continuation payoffs for the rational type of Player 1 are set so that she is willing to mix between different commitment actions.

- In an earlier draft of this paper, I show that the conclusion of Theorem 2 continues to hold when mixed commitment strategies are allowed, if another assumption, in addition to Assumption 1, is made to guarantee that the rational type of Player 1 can always reveal her type.[19] The problem becomes complicated in this case.[20] The definition of commitment

[19] Formally, this is equivalent to assuming that the commitment strategies do not have full support at any information set.

[20] However,in this case it is unnecessary to deal with the issue of multiple commitment types explicitly because from

payoff needs to be generalized, and the equilibrium strategies of the rational type of Player 1 and Player 2 involve mixed actions. Nonetheless, the basic idea of the proof in this case is the same as that of Theorem 2.

- The Perfect Folk Theorem is needed in the proof of Theorems 1 and 2 to guarantee that there exist subgame-perfect-Nash-equilibrium payoffs which support the equilibrium strategies during the screening phase. Although there is no reason to believe that the Perfect Folk Theorem will not hold when the players have different discount factors and mixed strategies are unobservable, this has not been proven.[21] This is the reason why Theorems 1 and 2, strictly speaking, only hold when the players are equally patient. The remaining parts of both proofs do not depend on the assumption of equally patient players. If the Perfect Folk Theorem holds in the case of comparably patient players, then Theorems 1 and 2 will apply to those cases as well.

5 Reputation Effects in Strongly-Dominant-Action Games

In this section I show that in strongly-dominant-action games, a player can build a reputation even when she is less patient than her opponent. Assume $y < 0$ and $x > 0$ in the stage-game depicted in Fig.6. In this game, the most favorable outcome for Player 1 is (A, A) which gives her a payoff

		Player 2 A	Player 2 B
Player 1	A	2,x	0,0
	B	0,0	y,2

Figure 6: A Strongly-Dominant-Action Game

of 2. If Player 1 is rational, she always chooses action A , since A is a strictly dominant action. If Player 2 is rational and knows that Player 1 is rational, he should also choose A. As a result, if the game is played once, it is very likely that the outcome will be (A, A). When the game is infinitely repeated and the players are patient enough, any average payoff profile that is strictly bigger than $(0, \frac{2x}{2+x})$ can be supported by a subgame-perfect Nash equilibrium. Theorem 4 below shows that

the perspective of Player 2, he is always facing one "aggregate" commitment type with mixed commitment strategies. For example, suppose there are two commitment types: ω_1 with probability p_1 and ω_2 with probability p_2; for Player 2, this is equivalent to facing a commitment type $\gamma_1 = \frac{p_1}{p_1+p_2}\omega_1 + \frac{p_2}{p_1+P_2}\omega_2$ with a probability of $p_1 + p_2$.

[21]Lehrer and Pauzner (1997) has shown that the folk theorem holds with different discount factors when mixed strategies are observable.

in this case, the folk theorem result is not robust to a small amount of private information. In particular, if there is a strictly positive probability that Player 1 is a commitment type who always chooses A, then (A, A) will be played along the equilibrium path in any perfect Bayesian equilibrium in the corresponding incomplete-information, infinitely-repeated game. This result critically depends on the assu pt on th t t e o ly om it en ty e f r P ay r 1 lw ys ho se A a d t at la er is at on l. ow ve , i do s n t r ly n t e a su pt on ha Pl ye 1 i in ni el mo e p ti nt ha Pl ye 2. n f ct th re ul ho ds ve wh n P ay r 1 s l ss at en th n P ay r 2

Th in ui io is s f ll ws Su po e t th co tr ry $(A\ A)$ s n t a wa s c os n o th eq i- ib iu pa h a d P ay r 1 s e ui ib iu pa off s s ri tl le s t an . I th t c se if la er ch os s A nd fin te y, he e m st e a er od n w ic Pl ye 2, cc rd ng o h s e ui ib iu st at gy wi l c oo e B it po it ve ro ab li y. in e a ti n B s n t a es re po se or la er to , P ay r 2 il ch os B o ly he he nt ci at s t at la er wi l c oo e B it po it ve ro ab li y i ei he th t p ri d o th fu ur . O th ot er an , S nc A i a s ri tl do in nt ct on or la er , s e w ll ho se in pe io as on as ha wi l n t a ve se y a ec he fu ur pa off Th t i , i is at on l f r P ay r 1 o c oo e B nl if la er wi l c oo e B n t e f tu e. n s or , P ay r 2 ho se B b ca se e t in s P ay r 1 il ch os B i th fu ur , a d P ay r 1 ho se B i th fu ur be au e s e t in s P ay r 2 il ch os B i th fu ur an so n. [2] H we er th s p oc ss an ot as in efi it ly f P ay r 2 pd te hi be ie ab ut la er 's yp ac or in to ay s' ul . U ti at ly th re xi ts pe io wh re it er la er or la er pl ys ub op im ll , v ol ti g a on it on or er ec Ba es an qu li ri m.

t i in er st ng o c nt as th ab ve as to he as wh re > 2 $y >$. I th la te ca e, he am is ne f c mm n- nt re t. ot pl ye s s ri tl pr fe th ou co e ($, A$ to ny th r, ut is o l ng r a tr ct y d mi an ac io fo Pl ye 1. in e t er is o c nfl ct f i te es s b tw en he wo la er , i ma ap ea th t t is s a it at on ar ic la ly on uc ve o r pu at on ev lo me t. t t rn ou th t c mm n- nt re t i ne th r a ec ss ry or uffi ie t c nd ti n f r t e e is en e o re ut ti n e ec s. ri ps nd ho as 97 sh w t at he mi ed tr te ie ar al ow d, ve if he e i a s al pr ba il ty ha Pl ye 1 i a c mm tm nt yp wh al ay ch os s A th re xi ts pe fe t B ye ia eq il br um it eq il br um ay ff f r P ay r 1

[2] N te ha it s i po ta t h re ha Pl ye 1 e pe ts la er to ho se st ic ly n t e f tu e.

arbitrarily close to her minmax payoff.[23] The sharp difference between these two cases illustrates the critical importance of the assumption that A is a strictly dominant action.[24] On the other hand, it is not important whether the game is of common interest, (i.e. $x > 2$), because Player 2 will choose A regardless of the value of x as long as he is convinced that Player 1 is going to choose A. Theorem 3 formalizes the above argument.

Theorem 6 *In an infinitely-repeated strongly-dominant-action game with one-sided incomplete information,* $\Gamma(A, g, \delta_1, \delta_2, \gamma_1, \mu_0^1)$*, if the commitment type,* γ_1*, always chooses the strongly-dominant action, then for any* $\delta_1, \delta_2 \in (0, 1)$*, and any* $\mu_0^1 > 0$*, the average equilibrium payoff for the rational type of Player 1 is equal to* $\bar{v}_1$ *in any perfect Bayesian equilibrium.*

Let $\hat{a}_1$ be the strongly-dominant action for Player 1 and $\hat{a}_2$ be one of Player 2's best responses to $\hat{a}_1$. Note that in this case, $\gamma_1(h_t) = \hat{a}_1,\ \forall h_t \in H$. Define $\hat{s}_2 \in S_2$ such that $\hat{s}_2(h_t) = \hat{a}_2,\ \forall h_t \in H$. For convenience, I use $\hat{v}_1$ and $\hat{v}_2$ to denote $g_1(\hat{a}_1, \hat{a}_2)$ and $g_2(\hat{a}_1, \hat{a}_2)$.

The idea of the proof is to show that if $(\sigma_1, \sigma_2, \mu)$ is a perfect Bayesian equilibrium in which the equilibrium payoff for Player 1 is strictly below $\bar{v}_1$, then it is possible to construct a history $\hat{h} \in H(\gamma_1)$ and a sequence of time $\{t_1, t_2, t_3, ...\}$ such that, for some $\epsilon \in (0, 1)$, $\mu(\hat{h}_{t_{i+1}}) \geq \frac{\mu(\hat{h}_{t_i})}{1-\epsilon}$ for all i. This implies that no matter how small μ_0^1, there exists a t_j such that $\mu(\hat{h}_{t_j}) > 1$, which contradicts the assumption that $\mu(h_t)$ □ $1 \quad \forall h_t \in H$. Thus, no such $(\sigma_1, \sigma_2, \mu)$ can exist.

Before I can construct $\hat{h} \in H(\gamma_1)$ and $\{t_1, t_2, t_3, ...\}$, I need to establish the following 3 lemmas. Lemma 5.1 below follows directly from the strict dominance of $\hat{a}_1$. If Player 1 knows that she can obtain $\bar{v}_1$ in the continuation game regardless of her current period's action, she will strictly prefer to choose $\hat{a}_1$ in that period.

Lemma 7 *If* $(\sigma_1, \sigma_2, \mu)$ *is a perfect Bayesian equilibrium of the game* Γ *and* $v_1(\gamma_1, \sigma_2 | \hat{a}_1) = \hat{v}_1$*, then* $\sigma_1(h_1) = \hat{a}_1$*.*

Lemma 5.2 is a common argument which asserts that when Player 2 discounts his future payoffs, there is a period N such that Player 2 cares very little about his payoffs after period N. To formally state Lemma 5.2, it will be convenient to define a N-period truncated strategy σ_i^N for any $\sigma_i \in \Sigma_i$.

[23] Aumann and Sorin (89) show that there exists a pure strategy equilibrium in this game, and all pure strategy equilibria are Pareto efficient.

[24] Cripps and Thomas (1997) assume that A is not a strictly dominant action.

F r a y $\sigma \in \Sigma$ an po it ve nt ge N, efi e σ_i a fo lo s:

$$
\begin{aligned}
{}_i^N\, h_t &= \sigma(h) \quad \forall \ \square N \quad 1 \\
\sigma_i\ (\ _t) &\quad a' \quad \forall t \quad N - \quad \text{fo so e } a_1 \in \ _1.
\end{aligned}
$$

${}_i^N$ s a ru ca ed er io of ${}_i$. t i eq iv le t t σ_i ur ng he rs $N-$ pe io s a d i eq iv le t t a c ns an st at gy ${}'_1$ ro pe io N o wa d. he ro fs f L mm s 5 2 a d 5 3 a e g ve in he pp nd x.

em a 8 *iv n* δ *,* $\forall \ > 0\ \exists N$ *uc th t* $\forall\ _1,\ _2$

$$|\ _2(\ _1,\ _2) \quad v_2\ \sigma^N, \sigma_2\ )\ \square\ \epsilon$$

Ne t, in ro uc a f nc io $d^N\ \rho_1\ : \Sigma \ \rightarrow \Re$ wh ch ea ur s t e " is an e" et ee $\rho_1 \quad \Sigma_1$ nd ${}_1$ i th fir t N 1 p ri ds[25] efi e

$$d\ \ (\rho\ ) = \max_{2 \in\ 2} (\ - P\ _{1,\ 2} (\ _N(\ _1,\ _2)\ )$$

w er $h_N\ \gamma_1\ s_2$ is he -1 er od is or in uc d b th co mi me t s ra eg γ_1 nd pu e s ra eg s_2 an $P_{\rho\ ,s}\ (h\ \ (\gamma\ , s\ ))$ s t e p ob bi it th t P ay r 1 bo h t e r ti na an co mi - m nt yp) c oo es a_1 or he rs N p ri d. hu , $d\ \ (\rho\ )$ i ma im m p ob bi it th t P ay r 1 il re ea he ty e i th fir t N 1 p ri ds ve al po si le tr te ie of la er . L mm 3 s ow th t i ρ_1 nd ${}_1$ a e " lo e" th n t ey en ra e s mi ar ay ffs or la er .

L mm 9 *G ve* $\delta_2\ \forall \xi \quad 0$ *a d p si iv in eg r N* $\exists \epsilon \quad 0$ *s ch ha if* ${}^N(\ _1) \quad \epsilon,$ *he* $\forall \sigma\ \in \Sigma$ *,*

$$|\ _2(\ _1^N\ \sigma^N) - \ _2(\ _1^N\ \sigma^N)| \quad \xi.$$

ro f o Th or m 3

Su po e ($_1$, $_2$,) i a p rf ct ay si n e ui ib iu su h t at he qu li ri m p yo fo Pl ye 1 i st ic ly es th n $\hat{\ }_1$. t f ll ws ha $v_1\ \gamma_1\ \sigma_2\ < \hat{\ }_1$, nd he e e is s s me is or $h_t \quad H(\ _1)$ n t e e ui ib iu pa h s ch ha $\sigma_2\ \hat{a}\ |h\ ) <\ $. I ot er or s, he e e is s s $s\ \in s\ pp\ \sigma_2\ h_t$ su h t at $_2(\ _1) \in b\ (\hat{\ }_1)$ Se $t_1 \quad t$ a d $h\ _1 = \ _t$, he

$$v_2\ \gamma_1\ s_2\ \square\ \hat{\ }_2 - \ , \text{ w er } c = 1 - \ _2)\ \hat{v}\ - \max_{a\ \in A\ /b\ (\hat{\ }_1)} \ _2(a_1\ a_2\ ).$$

y L mm 5. , s t N uc th t | $_2(\sigma_1\ \tilde{\sigma}\ ) - \ _2(\sigma^N, \tilde{\ }_2^N\ |\ \square\ \frac{}{8}\ \forall \sigma_1 \quad \Sigma_1\ \ \tilde{\sigma}\ \in \Sigma$. A a r su t, $_2(\ _1^N\ s^N)\ \square\ v_2 \quad \frac{7c}{}$.

[2] d^N .) ol ow fr m t e d fin ti n o cl se es be we n t o s ra eg in al i a d L hr r (3)

By Lemma 5.3, set ϵ such that if $d^N(\rho_1, \gamma_1) \square \epsilon$, then $|v_2(\rho_1^N, \tilde{\sigma}_2^N) - v_2(\gamma_1^N, \tilde{\sigma}_2^N)| \square \frac{c}{8}$, $\forall\ \tilde{\sigma}_2 \in \Sigma_2$. I shall show that it is impossible for $d^N(\rho_1, \gamma_1) \square \epsilon$. Suppose to the contrary, that $d^N(\rho_1, \gamma_1) \square \epsilon$, then

$$\begin{aligned} v_2(\rho_1^N, s_2^N) &\square v_2(\gamma_1^N, s_2^N) + \frac{c}{8} \\ \Rightarrow v_2(\rho_1^N, s_2^N) &\square \hat{v}_2 - \frac{6c}{8}. \end{aligned}$$

Since s_2 is a best response for Player 2,

$$\begin{aligned} v_2(\rho_1, s_2) &\geq v_2(\rho_1, \hat{s}_2) \\ \Rightarrow v_2(\rho_1^N, \hat{s}_2^N) &\square v_2(\rho_1^N, s_2^N) - \frac{c}{8} \\ \Rightarrow v_2(\rho_1^N, \hat{s}_2^N) &\square \hat{v}_2 - \frac{5c}{8}. \end{aligned}$$

Since $v_2(\gamma_1^N, \hat{s}_2^N) \geq \hat{v}_2 - \frac{c}{8}$,

$$|v_2(\rho_1^N, \hat{s}_2^N) - v_2(\gamma_1^N, \hat{s}_2^N)| \geq \frac{c}{2},$$

contradicting the assumption that $d^N(\rho_1, \gamma_1) \square \epsilon$. Put differently, in order to induce Player 2 to choose a strategy that is not a best response to γ_1, the rational Player 1 has to deviate from playing γ_1 with a probability bounded away from zero either in response to s_2 or in response to $\hat{s}_2$.

Let $\tilde{s}_2 \in S_2$ such that $1 - P_{\rho_1, \tilde{s}_2}(h_N(\gamma_1, \tilde{s}_2)) \geq \epsilon$. Along the path $h_N(\gamma_1, \tilde{s}_2)$, let $n \square N$ be the last period in which the rational Player 1 chooses $\hat{a}_1$ with probability strictly less than 1, so that $P_{\rho_1, \tilde{s}_2}(h_n(\gamma_1, \tilde{s}_2)) = P_{\rho_1, \tilde{s}_2}(h_N(\gamma_1, \tilde{s}_2))$.

Since Player 1 is willing to choose some action $a_1 \neq \hat{a}_1$, from Lemma 5.1, there exists a t'-period history $h_{t'} \in H(\gamma_1)$ such that Player 2 may choose some action $a_2 \neq \hat{a}_2$.

$$\text{Set} \quad t_2 = t_1 + n + t' \quad \text{and} \quad \hat{h}_{t_2} = \hat{h}_{t_1}.h_n(\gamma_1, \tilde{s}_2).h_{t'}.$$

Since $\hat{h}_{t_1}$, $h_n(\gamma_1, \tilde{s}_2)$, and $h_{t'}$ all belong to $H(\gamma_1)$, $\hat{h}_{t_2}$ belongs to $H(\gamma_1)$ as well. Moreover, according to Bayes' rule, $\mu(\hat{h}_{t_2}) \geq \frac{\mu(\hat{h}_{t_1})}{1-\epsilon}$. By repeating the above argument, I can construct $t_3, t_4, t_5 ...$ and $\hat{h}$ similarly, so that $\forall i,\ \mu(\hat{h}_{t_{i+1}}) \geq \frac{\mu(\hat{h}_{t_i})}{1-\epsilon}$. For any μ_0^1, there is a t_k such that $\mu(\hat{h}_{t_k}) > 1$; therefore, $(\sigma_1, \sigma_2, \mu)$ is not a perfect Bayesian equilibrium. $\square$

6 F lk he re s w th om it en Ty es

n t is ec io , I tr ng he th re ul s o Se ti n 4 n t o w ys Th or m 4 s a ol th or m w th ne si ed nc mp et in or at on It ho s t at or ny wo pe so in ni el -r pe te ga e w th ne si ed nc mp et in or at on hi h s ti fie th co di io s o ei he Th or ms or , a y p yo pr fil th t i in he nt ri r o th co ve hu l o th se of tr ct y i di id al y r ti na pa off ca be up or ed y a er ec Ba es an qu li ri m w en he la er ar su ci nt y p ti nt nd he ri r p ob bi it th t t e i fo me pl ye is co mi me t t pe s s fficen ly ma l. n o he word , w ca an on tr ct qu li ri m w ic gi es he at on l t pe f t e i fo me pl ye a l w p yo wi ho t a ec ing he ay ff f r t e u in or ed la er

Th or m 1 *If tw -p rs n i finte y- ep at d g me it on -s de in om le e i fo ma io* $\Gamma($ *, g* δ*,* $_1$*,* $^1_0$ *sa is es he on it on of it er he re 1 o 2, he* $\forall$ *(* $_1$*,* $_2$*) in (V),* $\underline{\delta}$ *a d* $^{-1}_0$ *su h t at* $\delta \geq \_$ *an* μ^1 □ $^{-1}_0$ *th re s a er ec Ba es an qu li ri m w th he at on l p ay rs eq il br um ay ffs qu l t (v , v).*

he de be in Th or m 4 s v ry im le Th or ms an 2 i pl th t t er ex st a p rf ct ay si n e ui ib iu in hi h t e e ui ib iu pa off or la er is $_1$; ow e w nt he ay ff f r P ay r 2 o b eq al o v .) he er ec Fo k T eo em mp ie th t t er ex st a s bg me pe fe t N sh qu li ri m i wh ch he qu li ri m p yo s f r P ay rs 1 and 2 are, respectively, v_1 and v_2. Now, I can construct an equilibrium in which if Player 1 chooses the commitment action in the first period, then the players will play the first equilibrium from period 2 onward; otherwise, they will play the second equilibrium. Since the rational type of Player 1 is indifferent between these two equilibria, it is rational for her to reveal her type with a sufficiently high probability such that the equilibrium payoff for Player 2 is approximately equal to v_2, his equilibrium payoff in the second equilibrium.

Proof of Theorem 4:

In period 1, the rational type of Player 1 chooses a_1^* with probability p_1, and imitates the commitment type by choosing $\gamma_1(h_1)$ with probability $1 - p_1$. Player 2 chooses some $\tilde{a_2}$ which is a best response to ρ_1, his expectation of Player 1's strategy in period 1. Define

$$\Lambda(a_1, a_2) = \frac{v_1 - (1-\delta) g_1(a_1, \tilde{a_2})}{\delta}.$$

The continuation payoffs from period 2 onward for the rational type of Player 1 and Player 2, $(v_1(a), v_2(a))$, and the beliefs about Player 1's type in period 2, $\mu(a)$, are given in Table 8.

$\mathbf{a}$	$\mathbf{v_1(a)}$	$\mathbf{v_2(a)}$	$\boldsymbol{\mu}\mathbf{(a)}$
$a_1 = \gamma_1(h_1)$	$\Lambda(a)$	$f(\Lambda(a))$	$\frac{\mu_0^1}{\mu_0^1+(1-\mu_0^1)(1-p_1)}$
$a_1 \neq \gamma_1(h_t)$	$\Lambda(a)$	$\frac{v_2-(1-\delta)g_2(\rho_1,\tilde{a_2})-(1-p_1+\mu_0^1 p_1)\delta f(\Lambda(a))}{(1-\mu_0^1)p_1}$	0

Table 8: The continuation payoffs, $(v_1(a), v_2(a))$, and beliefs , $\mu(a)$, in the second period

It is straightforward to verify that $\mu(a)$ is consistent with Bayes' rule and that given $(v_1(a), v_2(a))$, the first-period strategies of the players are rational, and the equilibrium payoffs for the entire game are equal to (v_1, v_2). To prove Theorem 4, I need to show that these continuation payoffs can be supported by some perfect Bayesian equilibrium when δ_1 is close enough to 1 and μ_0^1 is small enough.

Since $(v_1, v_2) \in int(V^*)$, it is clear that there are δ^* and μ^{1*} such that $\forall \delta \geq \delta^*, \quad \mu_1^0 \,\Box\, \mu^{1*}$, p_1 can be set so that $(v_1(a), v_2(a)) \in V^* \quad \forall a \in A$. In the case where $a_1 \neq \gamma_1(h_t)$, it follows from the Perfect Folk Theorem that there exists δ_1^{**} such that $\forall \delta_1 \geq \delta_1^{**}$, $(v_1(a), v_2(a))$ can be supported by some subgame-perfect Nash equilibrium. In the case where $a_1 = \gamma_1(h_1)$, Theorem 1 (or 2) implies that, in the game $\Gamma(A, g, \delta_1, \delta_2, \gamma_1 | a, \mu(a))$, there exist δ^{***} and μ^{1**} such that $\forall \delta \geq \delta^{***}$ and $\mu_0^1 \,\Box\, \mu^{1**}$ there is some perfect Bayesian equilibrium with equilibrium payoffs for the rational type of Player 1 and Player 2 equal to $\Lambda(a)$ and $f(\Lambda(a))$, respectively. Set $\mu_0^{1***} = \frac{(1-p_1)\mu^{1**}}{(1-p_1\mu^{1**})}$. It is straightforward to verify that Theorem 4 holds for all $\delta \geq \max\{\delta^*, \delta^{**}, \delta^{***}\}$ and $\mu_0^1 \,\Box\, \min\{\mu^{1*}, \mu^{1***}\}$. $\Box$

Theorem 5 extends the result of Theorem 4 to the case where both players possess private information. Formally, denote a two-person infinitely-repeated game with two-sided incomplete information by $\Gamma(A, g, \delta, \gamma_1, \mu_1^0, \gamma_2, \mu_0^2)$ where γ_2 and μ_0^2 represent the commitment type of Player 2 and the prior probability that Player 2 is a commitment type. (The other notation defined for the commitment type of Player 1 extends to that of Player 2.) Note that $\Gamma(A, g, \delta, \gamma_1, \mu_1^0, \gamma_2, \mu_0^2)$ can be reduced to a one-sided incomplete information game by setting either μ_0^1 or μ_0^2 to 0. Let Γ_i, for $i = 1, 2$, be the resulting repeated game in which only Player i possesses private information (i.e.

$\mu_0^j = 0,\ j \neq i$). Theorem 5 states that in any infinitely-repeated game $\Gamma(A, g, \delta, \gamma_1, \mu_1^0, \gamma_2, \mu_0^2)$, if Γ_1 and Γ_2 satisfy the conditions of Theorems 1 or 2, then any payoff profile that is in the interior of the set of strictly individually-rational payoff profiles can be supported by a perfect Bayesian equilibrium when the players are sufficiently patient, and the prior probability that the players are commitment types are sufficiently small. Theorem 5 demonstrates that the general result that reputation effects do not exist does not rely on the assumption of one-sided private information.

The basic idea of the proof is similar to that of Theorem 4. The key is to construct an equilibrium in which the rational types of both players will reveal their types with high probability in the first period. This is possible because when each player expects the other player to reveal her type, then each of them is basically choosing between playing a complete-information or a one-sided incomplete-information continuation game. If a player reveals her type, she will (most likely) play a perfect-information continuation game from period 2 onward; otherwise, she will play a continuation game in which her opponent is not sure about her type. Now from Theorem 4, I can construct continuation equilibria such that the rational type players are indifferent between imitating the commitment types and revealing their types.

Theorem 11 *In a two-person infinitely-repeated game with two-sided incomplete information $\Gamma(A, g, \delta, \gamma_1, \mu_0^1, \gamma_2, \mu_0^2)$, if both Γ_1 and Γ_2 satisfy the conditions of either Theorems 1 or 2, then $\forall (v_1, v_2) \in int(V^*)$, $\exists\ \underline{\delta}$, $\bar{\mu}_0^1$, and $\bar{\mu}_0^2$ such that $\forall \delta \geq \underline{\delta}$, $\mu_0^1 \Box \bar{\mu}_0^1$, and $\mu_0^2 \Box \bar{\mu}_0^2$, there is a perfect Bayesian equilibrium with the rational players' equilibrium payoffs equal to (v_1, v_2).*

Proof of Theorem 5:

First I establish a simple lemma which shows the existence of equilibrium. The proof is in the appendix. Let $(\tilde{v}_1, \tilde{v}_2)$ be the rational players' equilibrium payoff.

Lemma 12 *If $\Gamma(A, g, \delta, \gamma_1, \mu_1^0, \gamma_2, \mu_0^2)$ satisfies the conditions in Theorem 4, then $\exists\ \underline{\delta}$, $\bar{\mu}_0^1$, and $\bar{\mu}_0^2$ such that $\forall \delta \geq \underline{\delta}$, $\mu_0^1 \Box \bar{\mu}_0^1$, and $\mu_0^2 \Box \bar{\mu}_0^2$, a perfect Bayesian equilibrium exists.*

In period 1, the rational type of Player i chooses a_i^* with probability p_i and $\gamma_i(h_i)$ with probability $1 - p_i$. Consider Player 1's decision, if she chooses a_1^* in period 1. In period 2, she will play either a complete-information continuation game with probability p_2 or continuation game with her uncertain about Player 2's type with probability $1 - p_2$. On the other hand, if Player 1 chooses $\gamma_1(h_1)$ in period 1, then in period 2, she will play a continuation game with Player 2 uncertain about her type with probability p_2, and a two-sided incomplete-information continuation game with probability $1 - p_2$. The situation is similar for Player 2.

$$\text{Define} \quad \Lambda_1(a_1, a_2) = \frac{v_1 - (1-\delta) g_1(a_1, \rho_2)}{\delta_1},$$

$$\text{and} \quad \Lambda_2(a_1, a_2) = \frac{v_2 - (1-\delta) g_2(\rho_1, a_2)}{\delta_2}.$$

$\mathbf{a_1, a_2}$	$\mathbf{v_1(a)}$	$\mathbf{v_2(a)}$	$\boldsymbol{\mu_1}\mathbf{(a)}$	$\boldsymbol{\mu_2}\mathbf{(a)}$
$a_1 \neq \gamma_1(h_1)$ $a_2 \neq \gamma_2(h_1)$	$\Lambda_1(a)$	$\Lambda_2(a)$	0	0
$a_1 = \gamma_1(h_1)$ $a_2 \neq \gamma_2(h_1)$	$\frac{\Lambda_1(a) - (1-p_2+\mu_0^2 p_2)\tilde{v_1}}{(1-\mu_0^2)p_2}$	$\Lambda_2(a)$	$\frac{\mu_0^1}{\mu_0^1+(1-\mu_0^1)(1-p_1)}$	0
$a_1 \neq \gamma_1(h_t)$ $a_2 = \gamma_2(h_1)$	$\Lambda_1(a)$	$\frac{\Lambda_2(a) - (1-p_1+\mu_0^1 p_1)\tilde{v_2}}{(1-\mu_0^1)p_1}$	0	$\frac{\mu_0^2}{\mu_0^2+(1-\mu_0^2)(1-p_2)}$
$a_1 = \gamma_1(h_1)$ $a_2 = \gamma_2(h_1)$	$\tilde{v_1}$	$\tilde{v_2}$	$\frac{\mu_0^1}{\mu_0^1+(1-\mu_0^1)(1-p_1)}$	$\frac{\mu_0^2}{\mu_0^2+(1-\mu_0^2)(1-p_2)}$

Table 9: Continuation Payoffs, $v_1(a), v_2(a)$ in period 2

The continuation payoffs are set so that the equilibrium payoff for the rational type players in the game is equal to v_1 and v_2. It is straightforward to verify that $\mu_1(a)$ and $\mu_2(a)$ are consistent with Bayes' rule and that the first period strategies are rational given the beliefs and continuation payoffs. Using an argument similar to that of Theorem 4, it can be shown that $\exists\ \underline{\delta},\ \bar{\mu}_0^1$, and $\bar{\mu}_0^2$ such that $\forall \delta \geq \underline{\delta},\ \mu_0^1 \,\square\, \bar{\mu}_0^1$, and $\mu_0^2 \,\square\, \bar{\mu}_0^2$, p_1 and p_2 can be set such that the continuation payoffs $v_1(a),\ v_2(a)$ can be supported by perfect Bayesian equilibrium. $\square$

7 Conclusion

In this paper, I study a two-person infinitely-repeated game in which an informed player tries to develop a reputation by mimicking a commitment type, but her opponent, expecting that, tries to screen her out by choosing an action that will "punish" her for imitating the commitment type. Fudenberg and Levine (1989) and Schmidt (1993) point out that since screening is also costly to the uninformed player, he will screen only when he expects the informed player to reveal her type in the future with a probability bounded away from zero. When he sees the informed player choose the commitment action in a period in which he thinks that she may reveal her type, he assigns a higher probability to the event that she really is a commitment type. Based on this argument, they conclude that for any fixed discount factor of the uninformed player, the number of screening periods must be bounded. Hence, no matter how small the prior probability that the informed player is a commitment type, if she is sufficiently more patient than the uninformed player, she can guarantee herself a payoff strictly higher than the minimum payoff she may otherwise receive in the case of perfect information. This result, while very robust in many other aspects, obviously depends heavily on the assumption that the informed player is arbitrarily more patient than her opponent.

The main contribution of this paper is to show that, in general, when the two players are comparably patient, the informed player does not always benefit from developing a reputation. Formally, when the commitment strategy is simple and the stage game is neither a strongly-conflicting-interest game nor a strongly-dominant-action game, or when the commitment payoff is strictly less than the informed player's highest repeated-game payoff, any payoff that is in the interior of the set of strictly individually rational payoffs can be supported by some perfect Bayesian equilibrium as the discount factor goes to one and the prior probability that the informed player is a commitment type goes to zero. This result is very robust. With minor qualifications, it can be extended to allow for multiple commitment types, mixed commitment strategies, comparably patient players and two-sided private information. In other words, the folk theorem holds even when there is a small probability that the players are commitment types.

The fundamental idea is that the strength of reputation effects depend on the level of relative, but not absolute, patience. To clarify the relationship between reputation effects and relative patience, I introduce a measure of relative patience based on the weights the players put on their payoffs after a certain future period. Absolute patience does not matter because as the informed player becomes more patient, the uninformed player will also become more patient and, hence, may screen for a longer period of time. For a fixed level of relative patience, if the prior probability that the informed player is a commitment type is sufficiently small, the uninformed player can be induced to screen for such a long period that it is not worthwhile for the informed player to develop a reputation.

Another crucial aspect of the proofs of Theorem 1 and 2 is that the informed player can be induced to mix between mimicking the commitment type and revealing her type. Giving up her reputation is not necessarily bad for a rational informed player because, according to the Perfect Folk Theorem, any strictly individually rational payoff can be supported by a subgame-perfect Nash equilibrium. As a result, with the exception of strongly-dominant-action games, there always exist continuation equilibrium payoffs such that the rational informed player is indifferent between mimicking the commitment type and revealing her type. The importance of this point is demonstrated by the example of infinitely-repeated strongly-dominant-action games. In such games, if the only commitment type is one who always chooses the strongly-dominant action, the rational informed player cannot be induced to reveal her type in the last period of screening. In that case, I show that the informed player receives the commitment payoff in all perfect Bayesian equilibria.

Finally, I would like to remind the readers that Assumption 1 is restrictive. It rules out a commitment payoff equal to the highest subgame-perfect-Nash-equilibrium payoff for the informed player. From Theorem 1, these commitment types are likely to be the most conducive to the development of reputation. When the commitment strategy is history-dependent, the action of the uninformed player in one period will have long-term payoff consequences due to its effect on the future actions of the commitment type. In my current approach, these effects are balanced off by adjusting the informed player's continuation payoffs on equilibrium paths (after some deviations by the uninformed player). This, in turn, requires that the continuation payoff for the informed player not be an extremal point of the payoff set. At this point, it is not clear to what extent Assumption 1 can be weakened. I intend to clarify this question in future work.

8 Appendix

Proof of Lemma 2.1:

Suppose (1) is true, given any $\epsilon, \eta > 0$, $\exists$ $\underline{\delta}_1$ s.t. $\forall \delta_1 \geq \underline{\delta}_1$

$$0 \Box \frac{\ln \delta_1}{\ln \delta_2(\delta_1)} \Box \frac{\ln(1-\eta)}{\ln \epsilon}$$

Since the last expression can be made to be arbitrarily small by pushing η to zero, $\lim_{\delta_1 \to 1} m(\delta_1, \delta_2(\delta_1))$ exists and is equal to 0.

To show the converse is true, suppose $\lim_{\delta_1 \to 1} m(\delta_1, \delta_2(\delta_1)) = 0$, then

$$\forall \gamma_1 > 0 \; \exists \underline{\delta}_1 \text{ s.t. } \forall \delta_1 \geq \underline{\delta}_1, \quad \frac{\ln \delta_1}{\ln \delta_2(\delta_1)} \Box \gamma_1$$

For arbitrary ϵ, $\eta \in (0,1)$, set $\gamma_1 = \frac{\ln(1-\eta)}{\ln \epsilon}$, and (1) follows. $\square$

Proof of Lemma 2.2:

1.

$$\begin{aligned} \delta_2(\delta_1)^{T(\delta_1)} &= \delta_2(\delta_1)^{\square \frac{\ln(\frac{\mu_0}{\bar{\mu}})}{\ln(1-\Delta(\delta_1))} + [T(\delta_1) - \frac{\ln(\frac{\mu_0}{\bar{\mu}})}{\ln(1-\Delta(\delta_1))}]\Big]} \\ &= \delta_2(\delta_1)^{[T(\delta_1) - \frac{\ln(\frac{\mu_0}{\bar{\mu}})}{\ln(1-\Delta(\delta_1))}]} . \frac{\mu_0}{\bar{\mu}}^{\frac{\ln \delta_2(\delta_1)}{\ln(1-\Delta(\delta_1))}} . \end{aligned}$$

Since $\lim_{\delta_1 \to 1}(T(\delta_1) - \frac{\ln(\frac{\mu_0}{\bar{\mu}})}{\ln(1-\Delta(\delta_1))}) = 0$, therefore $\lim_{\delta_1 \to 1} \delta_2(\delta_1)^{[T(\delta_1) - \frac{\ln(\frac{\mu_0}{\bar{\mu}})}{\ln(1-\Delta(\delta_1))}]} = 1$.

Next, the limit of the exponent of the second term as δ_1 goes to 1, which by L'Hospital rule, is given by:

$$\lim_{\delta_1 \to 1} \frac{\ln \delta_2(\delta_1)}{\ln(1-\Delta(\delta_1))} = \frac{q}{d}$$

It follows that

$$\begin{aligned} \lim_{\delta_1 \to 1} \delta_2(\delta_1)^{T(\delta_1)} &= \lim_{\delta_1 \to 1} \delta_1^{[T(\delta_1) - \frac{\ln(\frac{\mu_0}{\bar{\mu}})}{\ln(1-\Delta(\delta_1))}]} \lim_{\delta_1 \to 1} (\frac{\mu_0}{\bar{\mu}})^{\frac{\ln \delta_2(\delta_1)}{\ln(1-\Delta(\delta_1))}} \\ &= (\frac{\mu_0}{\bar{\mu}})^{\frac{q}{d}} \end{aligned}$$

$\square$

2.

$$\begin{aligned} \lim_{\delta_1 \to 1} \delta_1^{T(\delta_1)} &= \lim_{\delta_1 \to 1} \delta_2(\delta_1)^{T(\delta_1) \frac{\ln \delta_1}{\ln \delta_2(\delta_1)}} \\ &= \lim_{\delta_1 \to 1} \delta_2(\delta_1)^{T(\delta_1)(\lim_{\delta_1 \to 1} \frac{\ln \delta_1}{\ln \delta_2(\delta_1)})} \\ &= \lim_{\delta_1 \to 1} \delta_2(\delta_1)^{T(\delta_1) m} \\ &= \frac{\mu_0}{\bar{\mu}}^{\frac{mq}{d}} \end{aligned}$$

$\square$

Proof of lemma 2.3:

Given μ_0 and $\frac{\epsilon}{2}$, by Lemma 2.2 $\exists$ $\underline{\delta_1}$ such that $\forall$ $\delta_1 \geq \underline{\delta_1}$, $\delta_1^{T(\delta_1)}$ $\square$ $\left(\frac{\mu_0}{\bar{\mu}}\right)^{\frac{\bar{m}q}{d}} + \frac{\epsilon}{2}$. Set $\bar{\mu_0}$ such that $\left(\frac{\mu_0}{\bar{\mu}}\right)^{\frac{\bar{m}q}{d}} = \frac{\epsilon}{2}$. Since $\left(\frac{\mu_0}{\bar{\mu}}\right)^{x}$ is strictly increasing in $\frac{\mu_0}{\bar{\mu}}$ when x is positive, $\forall$ μ_0 $\square$ $\bar{\mu}_0$ and

$\delta_1 \geq \bar{\delta_1},\ \delta_1^{T(\delta_1)} \Box \epsilon.$ $\square$

Proof of Lemma 5.2:

Since σ_1 , and σ_2 are identical to σ_1^N and σ_2^N in the first N-1 periods; therefore

$$|v_2(\sigma_1, \sigma_2) - v_2(\sigma_1^N, \sigma_2^N)| \Box \delta_2^N d$$

The desired inequality is obtained by setting N to be bigger than $\frac{\ln(\epsilon/d)}{\ln \delta_2}$. $\square$

Proof of Lemma 5.3:

First notice that

$$\begin{aligned} v_2(\rho_1^N, \sigma_2^N) &= E_{\sigma_2}[v_2(\rho_1^N, s_2^N)] \\ &= |E_{\sigma_2}[\{1 - P_{\rho_1, s_2}(h_N(\hat{s}_1, s_2))\}E_{\rho_1}(v_2(s_1^N, s_2^N|s_1 \neq \hat{s}_1)) + P_{\rho_1, s_2} v_2(\hat{s}_1^N, s_2^N)]| \end{aligned}$$

As a result,

$$\begin{aligned} &|v_2((\rho_1^N, \sigma_2^N) - v_2(\hat{s}_1^N, \sigma_2^N)| \\ = |E_{\sigma_2}[(1 - P_{\rho_1, s_2}(h_N(\hat{s}_1, s_2)))&E_{\rho_1}(v_2(s_1^N, s_2^N|s_1 \neq \hat{s}_1)) - v_2(\hat{s}_1^N, s_2^N)]| \\ &\Box \epsilon|E_{\sigma_2}[E_{\rho_1}(v_2(s_1^N, s_2^N|s_1 \neq \hat{s}_1)) - v_2(\hat{s}_1^N, s_2^N)]| \\ &\Box \epsilon(1 - \delta_2^N)d \end{aligned}$$

The desired inequality can be obtained by setting $\epsilon = \frac{\xi}{(1-\delta_2^N)d}$. $\square$

Proof of Lemma 6.1:

By definition, $\exists v_2 \in V_2^*$ s.t. $(\nu(\gamma_1), v_2) \in V^*$. It is straightforward to construct an equilibrium with the following features: In period 1 the rational type of Player 1 reveals her type and the rational type of Player 2 chooses the commitment action. In period 2, from Theorem 4, construct a perfect Bayesian equilibrium for the one-sided incomplete-information continuation game with payoffs such that the payoff for the entire game is equal to $(\nu(\gamma_1), v_2)$. Note that Player 1 is willing to reveal her type in period 1 because she is getting her commitment payoff. $\square$

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