Exact Arbitrage and Portfolio Analysis in Large Asset Markets*

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Summary. We provide a detailed portfolio analysis for a financial market with an atomless continuum of assets. In the context of an exact arbitrage pricing theory (EAPT), we go beyond the characterization of the existence of important portfolios (normalized riskless, mean, cost, factor and mean-variance efficient portfolios) to furnish exact portfolio compositions in terms of explicit portfolio weights. Such an analysis has not been furnished before in the context of the asymptotic arbitrage pricing theory (APT). We also characterize conditions under which a mean-variance efficient portfolio is a benchmark portfolio used in the EAPT to proxy essential risk. We illustrate our results with several examples of specific financial markets.

Keywords and Phrases: exact arbitrage, portfolio weights, well-diversified portfolio, mean-variance efficient portfolio, mean, cost and factor portfolios, Loeb measure space.

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1 Introduction

The fundamental rationale for the consideration of infinite asset markets in financial economics is to provide an analytically viable formalization of a well-diversified portfolio, one which contains only factor, as opposed to idiosyncratic, variance. The theoretical task is to devise an explanation for the intuition that under certain no arbitrage conditions, the risk premium of an asset depends in a linear form only on systematic or factor risk. If the market is in equilibrium in the sense that there is no possibility of gains from arbitrage, it will not reward any risk that can be (naively) diversified away. The original heuristic arguments of Ross are phrased in terms of portfolios consisting of infinitesimal amounts of an infinite number of assets, the so-called arbitrage portfolios, and appeal both to the absence of arbitrage opportunities and to the classical law of large numbers to provide an asset pricing formula in which idiosyncratic risk is (approximately) eliminated; see [24, pp. 196-199] and [25, pp. 67-68]. Such an argument cannot be executed in a finite universe of assets.

There have been two different approaches to address these issues and to formalize Ross' heuristics. The first is to consider markets with countable many financial assets, each of whose one-period returns are assumed to have a finite variance and hence to lie in a Hilbert space $L^2(\Omega)$ of square integrable functions on a sample space Ω; see [23, 24, 25], [6], [7], among others.[1] In such a setting, the rate of return to any finite portfolio also lies in $L^2(\Omega)$ and the weight of each asset in the portfolio's composition is well-specified and clear. However, barring trivial cases, no finite portfolio can be well-diversified, and the basic idea is to bypass this issue by remaining in the (Hilbert) space of portfolio returns, and by defining a well-diversified portfolio as the limit, in the Hilbert space norm, of the *returns* of a sequence of finite portfolios. In short, to address the problem by a procedure that blurs the very distinction between the return and the composition of an infinite portfolio. An appeal to a version of the classical law of large numbers then delivers a cancellation of the idiosyncratic risk in such a well-diversified portfolio. The same approach can be easily applied to markets with an asset index set of an *arbitrary* infinite cardinality under the assumption of no asymptotic arbitrage.[2]

The other approach is to consider markets with a continuum of assets with a special measure-theoretic structure in which an *exact* law of large numbers for a continuum of random variables allows the complete cancellation of idiosyncratic risk [28, 29]. In this continuum setting, portfolios are identified as the square integrable functions on the space T of asset names, i.e., as elements of $L^2(T)$, and the process of asset returns x is assumed to be an element of the parent Hilbert space $L^2(T \times \Omega)$. The emphasis on the variance-covariance matrix, as in [6, 7], is transferred to an emphasis on the autocorrelation function of asset returns, and its denumerable infinity of eigenfunctions are identified as factor loadings lying in one Hilbert space $L^2(T)$, and these in turn determine the corresponding factors in another Hilbert space $L^2(\Omega)$.[3] The exact law of large numbers then says that *all* portfolios as defined this way have only factor risk and are therefore well-diversified automatically. In this concrete context of an idealized limit model, one can forego asymptotics, and formulate a notion of the absence of arbitrage opportunities in an exact, rather than asymptotic, form, and develop an *exact* arbitrage pricing theory, termed

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[1] For a detailed discussion of earlier work, see [15, 16] and their references.

[2] See [15] and its references.

[3] For this type of structural result, see [28, 29] and Section 2 below.

EAPT in [16].

Thus, both approaches formalize the cancellation of idiosyncratic risk and deliver an asset pricing formula under the assumption of the absence of arbitrage opportunities, asymptotic in one case and exact in the other.[4] There are important differences in detail however: the notion of exact arbitrage in the EAPT is both necessary and sufficient for the factor pricing formula to hold, and this formula implies and is implied by a beta pricing formula in which the denumerable sources of factor risk in the factor pricing formula are collapsed into an identifiable single source – a component of systematic risk termed essential risk – and the beta of its corresponding benchmark index portfolio measures the risk premium of almost all assets. But at this stage, a more relevant issue is whether we can go beyond an asset pricing formula and present a more detailed portfolio analysis for financial markets where idiosyncratic risk is exactly cancelled in well-diversified portfolios. Such an analysis ought to address a variety of outstanding and important questions: Can a normalized riskless portfolio be identified in an ensemble of risky assets? What are the necessary and sufficient conditions for its existence? Do there exist portfolios which generate the factors? Do cost and mean portfolios exist? Do they generate the mean-variance efficient set? Can one compute the maximal (linear) tradeoff, namely, the maximal Sharpe measure, between the mean and standard deviation of an efficient portfolio?

All of these questions are answered here. Given the special measure-theoretic structure, portfolio composition in terms of relative weights of any two assets can be clearly specified,[5] and we can go beyond formal properties to provide explicit formulae for a detailed study of important portfolios. In contrast, such explicit portfolio weights cannot be given for a well-diversified portfolio under the asymptotic approach.[6] Thus, concrete formulae of mean-variance efficiency that are lost in the asymptotic setting are recovered in the continuum setting. The plan of the paper then is as follows: after a section on the mathematical underpinnings, and another on the basics of the model, normalized riskless and factor portfolios are considered in Section 4, mean and cost portfolios in Section 5, and mean-variance efficient portfolios in Section 6. We illustrate the conditions of the theorems in Sections 4 and 5 by examples of specific financial markets; see Table 1 for an overview. Section 7 concludes the paper with two additional remarks, and an appendix collects some technicalities for the convenience of the interested reader. These results, in terms of the formulae they provide, add a concrete and computational aspect to the EAPT, and to one-period factor pricing models more generally, that has not been reported before.

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[4] The adjectives “asymptotic” and “exact” apply to the cancellation of idiosyncratic risk, to the asset pricing formula and to the notion of arbitrage; compare [6, Theorem 1] and the theorems in Section 2 of [15] to [16, Theorem 1].

[5] Here, one can usefully think of Aumann’s [5] rendering of the Debreu-Scarf theorem, and note that the question of working with finite types does not arise in an investigation whose primary motivation is the formalization of infinite diversification and diffuseness in an idealized setting.

[6] In the simplest case where one takes $1/n$ each for the first n assets in a market with an infinite sequence of assets, the limit portfolio corresponds to the density charge, a purely finitely additive measure; see [8, Exercise 9.8] and [32]. Portfolios based on purely finitely additive measures on the set $\mathbb{N}$ of natural numbers do not give any relative portfolio weights explicitly as is done in this paper. Also, it is shown in [31] that statements on diversification in the framework of purely finitely additive measures may not be reliable in the sense that arbitrary results can be obtained.

2 Mathematical Preliminaries and a Portmanteau Theorem

The model and results presented in this paper draw on vocabulary from four mathematical sources: (i) elements of the theory of Hilbert spaces as presented in [27, Chapter 4] and in [10];[7] (ii) integration theory, specifically integration on product measures, as presented in [19, Chapter VIII] or [27, Chapter 7]; (iii) probability and statistics, specifically principal components and factor analysis,[8] as presented in [21] and [9]; (iv) the elements of nonstandard analysis, as presented in [18] and [11] and, for applications in mathematical economics, in [1, 2]. In the remainder of this section, we develop the terminology to state a portmanteau theorem that serves as a point of departure for our analysis.

Let $(T, \mathcal{T}, \lambda)$ be the Loeb counting probability space on a hyperfinite set T. Note that $(T, \mathcal{T}, \lambda)$ is an atomless measure space constructed from an internal counting probability space on $(T, \bar{\mathcal{T}}, \bar{\lambda})$. It is of course not necessary to use the counting measure on T; it is simply a natural one given our interest in the relative portfolio weights of assets.[9] Let $(\Omega, \mathcal{A}, P)$ be another atomless Loeb measure (probability) space. As is by now well-understood in the economics literature, Loeb measure spaces, even though constituted by nonstandard entities, are standard measure spaces in the specific sense that any result proved for an abstract measure space applies to them. For details regarding both of the above two statements, see Anderson [1, 2] and his references.

The usual product space is denoted by $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \otimes P)$. However, there is another product space $(T \times \Omega, \mathcal{T} \otimes^L \mathcal{A}, \lambda \otimes^L P)$, called the Loeb product space, that extends the usual product, retains the Fubini property and is rich enough for the study of a continuum of independent random variables; see [29]. We begin by noting the interesting fact that the completion of the standard product measure space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \otimes P)$, corresponding to the standard measure spaces $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{A}, P)$, is always strictly contained in the Loeb product space $(T \times \Omega, \mathcal{T} \otimes^L \mathcal{A}, \lambda \otimes P)$; see [29, Proposition 6.6]. For simplicity, let $\mathcal{U}$ denote the product σ-algebra $\mathcal{T} \otimes \mathcal{A}$. Since the measure $\lambda \otimes^L P$ is an extension of $\lambda \otimes P$ on the usual product σ-algebra $\mathcal{T} \otimes \mathcal{A}$ to the larger product σ-algebra $\mathcal{T} \otimes^L \mathcal{A}$, we use $\lambda \otimes P$ to replace $\lambda \otimes^L P$ in the sequel, also for notational simplicity. We emphasize that we always work with the larger product σ-algebra $\mathcal{T} \otimes^L \mathcal{A}$.

For an integrable real-valued process g on the Loeb product space, let $E(g|_{\mathcal{U}})$ denote the conditional expectation of g with respect to $\mathcal{U}$; see [20, Chapter VIII] for details as to conditional expectations. This conditional expectation is a key operation, introduced in [28]. In our context, such an operation involves both a product σ-algebra $\mathcal{U}$ and a natural but significant extension of it, the Loeb product algebra $\mathcal{T} \otimes^L \mathcal{A}$. As such, it has no natural counterpart in standard mathematical practice or in nonstandard mathematics using only internal entities; see [28, 29] for details.

As is usual, we shall refer to a measurable function of two variables as a process. Given

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[7] The importance of the projection theorem and the Reisz representation theorem for the subject matter of this paper has already been stressed; in particular, projection maps on Hilbert spaces have played a fundamental role in [6], [7] and [15]. Here we shall also need some basic properties of compact operators on a Hilbert space.

[8] The relevance of these subjects has been discussed in some detail in [6] and in [7], and undoubtedly in an extensive empirical literature. In this connection, also see [30].

[9] A common infinitesimal unit allows a direct comparison; see the discussion in the paragraph below Definition 1.

a process g on the Loeb product space, for each $t \in T$, g_t denotes the function $g(t,\cdot)$ on Ω, and for each $\omega \in \Omega$, g_ω denotes the function $g(\cdot,\omega)$ on T. The functions g_t are usually called the random variables of the process g, while the g_ω are referred to as the sample functions of the process. Note that the measurability of g_t and g_ω is a simple consequence of a Fubini type result for Loeb product measures, also referred to as Keisler's Fubini theorem [18]. One can understand a Fubini-type result on iterated integrals in the hyperfinite Loeb measure setting as the simple observation that hyperfinite sums can be exchanged. Hereafter, when the need arises, we simply change the order of integrals without an explicit statement on the application of any Fubini-type results.

We shall be working in a Hilbert space $L^2(T \times \Omega)$ of square integrable functions on the *bigger* product of two atomless Loeb probability spaces, the Loeb product space. Thus, for any $x \in L^2(T \times \Omega)$,

$$\int\int_{T\times\Omega} x^2(t,\omega)d\lambda \otimes P(t,\omega) \ < \ \infty. \tag{1}$$

We shall use the notation $(\cdot,\cdot)$ to denote the inner products in the Hilbert spaces $L^2(T)$ and $L^2(\Omega)$. Thus, for example, for any $p,q \in L^2(T)$, $(p,q) = \int_T p(t)q(t)d\lambda$. For any $x \in L^2(T \times \Omega)$, let μ be the mean function of the random variables embodied in the process x, which is to say that for any $t \in T$, $\mu(t) = \int_\Omega x(t,\omega)dP(\omega)$. It is clear that x is also $\lambda \otimes P$-integrable. An appeal to Keisler's Fubini-type theorem for Loeb measures then guarantees that μ is a Loeb integrable function on $(T, \mathcal{T}, \lambda)$. In fact, by the Cauchy-Schwarz inequality, it is clear that

$$\int_T \mu^2(t)d\lambda \leq \int\int_{T\times\Omega} x^2(t,\omega)d\lambda \otimes P(t,\omega) < \infty, \tag{2}$$

and hence μ is λ-square integrable and belongs to the Hilbert space $L^2(T)$. In the sequel, $\mu(t)$ will also be denoted by μ_t. A random variable with zero mean and a process with a null mean function will be referred to as a centered random variable and a centered process respectively.

We now collect for the reader's convenience relevant results from [29, Corollary 4.8]), [28, Theorems 1-3], as a portmanteau theorem.

Theorem A: *Let f be a real-valued square integrable centered process on the Loeb product space $(T \times \Omega, \mathcal{T} \otimes^L \mathcal{A}, \lambda \otimes P)$. Then f has the following expression: for $\lambda \otimes P$-almost all (t,ω) in $T \times \Omega$,*

$$f(t,\omega) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t)\varphi_n(\omega) + e(t,\omega),$$

with properties:
(i) $\lambda_n, 1 \leq n < \infty$ is a decreasing sequence of positive numbers; the collection $\{\psi_n : 1 \leq n < \infty\}$ is orthonormal; and $\{\varphi_n : 1 \leq n < \infty\}$ is a collection of orthonormal and centered random variables.
(ii) $E(f|_\mathcal{U})(t,\omega) = \sum_{n=1}^{\infty} \lambda_n\psi_n(t)\varphi_n(\omega)$ and $E(e|_\mathcal{U}) = 0$.
(iii) The random variables e_t are almost surely orthogonal, which is to say that for $\lambda \otimes \lambda$-almost all $(t_1,t_2) \in T \times T$,

$$\int_\Omega e_{t_1}(\omega)e_{t_2}(\omega)dP(\omega) = 0.$$

(iv) If $p \in L^2(T)$, then for P-almost all $\omega \in \Omega$,

$$\int_T p(t)e_\omega(t)d\lambda(t) = 0 \text{ and } \int_T p(t)f(t,\omega)d\lambda(t) = \sum_{n=1}^{\infty} \lambda_n \left(\int_T p(t)\psi_n(t)d\lambda(t)\right) \varphi_n(\omega).$$

(v) If $\alpha \in L^2(\Omega)$, then for λ-almost all $t \in T$, it is orthogonal to e_t, and

$$\int_\Omega \alpha(\omega)f(t,\omega)dP(\omega) = \sum_{n=1}^{\infty} \lambda_n \left(\int_\Omega \alpha(\omega)\varphi_n(\omega)dP(\omega)\right) \psi_n(t).$$

The structural result in Theorem A can be seen as a hyperfinite version[10] of the classical factor model with a finite population: the centered random variables φ_n as factors, the corresponding functions ψ_n as factor loadings, and the decreasing sequence of numbers λ_n as scaling constants, with the size of λ_n measures the role of the factor φ_n in understanding the correlational structure of f. It is worth emphasizing that the factors are endogenously derived by virtue of the fact that the associated autocorrelation function of the process f, $R(t_1,t_2) = \int_\Omega f(t_1,\omega)f(t_2,\omega)dP$, by serving as a kernel, defines an integral operator K on the space $L^2(T)$. That is, $K(p)(t_1) = \int_T R(t_1,t_2)p(t_2)d\lambda(t_2)$ for $p \in L^2(T)$. It is easily checked that λ_n^2 is in fact the n-th positive eigenvalue of the operator K with eigenfunction ψ_n, with all the eigenvalues listed according to reverse order and repeated up to their corresponding multiplicities. Note that if there are only m positive eigenvalues, then the infinite sum in Theorem A should be read in the sequel as a finite sum of m terms. It is also clear that $\varphi_n(\omega) = (1/\lambda_n)\int_T f_\omega(t)\psi_n(t)d\lambda$.

Conditions (i) and (ii) say that the conditional expectation $E(f|_\mathcal{U})$ has a biorthogonal expansion in which both the random variables φ_n and the functions ψ_n are orthogonal among themselves. The corresponding continuous analogue for processes which are continuous in quadratic means on an interval is often called the Karhunen-Loéve expansion theorem and is well-known; see [20]. Note that it is a trivial matter to require the factors to be orthonormal, but non-trivial to show that both factors and factor loadings can be orthogonal among themselves; see [28, 29] for details.

3 The Model and Two Benchmark Results

We now have all that we need to present the basic ingredients of the exact arbitrage asset pricing theory EAPT, rudiments of which are presented in [13] and [16]. The space $(T,\mathcal{T},\lambda)$ is to be used as the index set of asset names, and the space $(\Omega,\mathcal{A},P)$ as the sample space, one that formalizes all possible uncertain social or natural states relevant to the asset market. The financial market (henceforth, also referred to simply as a "market") is a real-valued $\mathcal{T} \otimes^L \mathcal{A}$-measurable function x on $T \times \Omega$, and the real-valued random variable x_t is interpreted as the one-period random return to an asset t in T. The assumption that x is in $L^2(T \times \Omega)$ guarantees that the asset return process has a second moment. Let μ be the mean function defined in

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[10] For the finite classical version, see [9], [21]; also [30].

the previous section. The centered process f, defined by $f(t,\omega) = x(t,\omega) - \mu(t)$, embodies the unexpected or the net random return of all the assets, and is also $\lambda \otimes P$-square integrable.

A portfolio is simply a function listing the amounts held of each asset. Since short sales are allowed, this function can take negative values. The cost of each asset is assumed to be unity, and hence the cost of a particular portfolio is simply its integral with respect to λ. Since we are interested in the mean and variance of the return realized from a portfolio, we shall assume it to be a square integrable function. The random return from a particular portfolio then depends on the random return, and the amounts held in the portfolio, of each asset $t \in T$. Formally,

Definition 1 *A portfolio is a square integrable function p on $(T, \mathcal{T}, \lambda)$. The cost $C(p)$ of a portfolio p is given by $(p, 1) = \int_T p(t) d\lambda(t)$. The random return of the portfolio p is given by $\mathcal{R}_p(\omega) = (p, x_\omega) = \int_T p(t)x(t,\omega)d\lambda(t)$. The mean (or the expected return) $E(p)$ and the variance $V(p)$ of the portfolio p are the mean and the variance of the random return $\mathcal{R}_p$ respectively.*[11]

Heuristically, $d\lambda(t)$ is interpreted as an infinitesimal amount of an asset t and can be regarded as a small accounting unit in some sense. Since $\bar{\lambda}$ is the counting probability measure on T,[12] all the assets have the same small accounting unit. Thus, in the portfolio p, $p(t)d\lambda(t)$ is the amount, and $p(t)x(t,\omega)d\lambda(t)$ is the return, of shares of asset $t \in T$. For any two assets, s and t in T, $p(s)$ and $p(t)$ measure their relative amounts in the portfolio p. Since a finite number multiplied by an infinitesimal is still an infinitesimal, the amount invested in, and the return pertaining to, any asset is infinitesimal, and this is the reason why any portfolio is well-diversified automatically.

Theorem A then formalizes the fact that a continuous ensemble of risk can be endogenously decomposed into two parts: a systematic (factor) part and a idiosyncratic (non-factor) part. It identifies denumerably-many sources of risk, φ_n, that "explain" the systematic part, and furthermore, ranks the relative importance of these sources by positive numbers λ_n. The dependence of an asset's rate of return on these sources is captured by the loading functions ψ_n. The residual to factor risk is idiosyncratic risk represented by e_t, which have no presence at all in well-diversified portfolios.

$$\begin{aligned} \mathcal{R}_p(\omega) &= \int_T p(t)\mu(t)d\lambda + \sum_{n=1}^{\infty} \lambda_n \left(\int_T p(t)\psi_n(t)d\lambda \right) \varphi_n(\omega) \\ &= (p,\mu) + \sum_{n=1}^{\infty} \lambda_n (p,\psi_n)\varphi_n(\omega). \end{aligned} \tag{3}$$

Hence, by the fact that φ_n, $n \geq 1$, are orthonormal with means zero,

$$E(p) = (p,\mu) = \int_T p(t)\mu(t)d\lambda; V(p) = \sum_{n=1}^{\infty} \lambda_n^2 (p,\psi_n)^2 = \sum_{n=1}^{\infty} \lambda_n^2 \left(\int_T p(t)\psi_n(t)d\lambda \right)^2. \tag{4}$$

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[11] Since $C(p)$ is the integral of the function p, one may tempt to think that is the mean of the portfolio p. However, one has to bear in mind that the mean of the portfolio p actually refers to the mean of the *random return* of the portfolio p.

[12] See the second paragraph of Section 2 above.

In particular, p is a riskless portfolio, which is to say that $V(p) = 0$, if and only if p is orthogonal to all of the ψ_n, to be expressed henceforth as $p \perp \psi_n$ for all $n \geq 1$.

Let $\mathcal{L}$ denote the closed subspace spanned by all of the ψ_n, and consider the portfolio h defined by projecting the constant function 1 on $\mathcal{L}^{\perp}$, the complementary subspace in $L^2(\Omega)$ orthogonal to $\mathcal{L}$. For each $n \geq 1$, let $s_n = (1, \psi_n) = \int_T \psi_n(t) d\lambda(t)$, and note that h is given by $h(t) = 1 - \sum_{n=1}^{\infty} s_n \psi_n(t)$. Since $h \perp \psi_n(\cdot)$ for each $n \geq 1$, h is a riskless portfolio. Due to its special nature, h will be called the *intrinsic riskless portfolio.* It is easy to see that

$$(h, h) = \int_T h^2(t) d\lambda = (h, 1) = \int_T h(t) d\lambda = C(h) \equiv h_0 = 1 - \sum_{n=1}^{\infty} s_n^2, \tag{5}$$

and hence $h \equiv 0$ if and only if $\sum_{n=1}^{\infty} s_n^2 = 1$.[13] We can now define a special parameter μ_0 by

$$\mu_0 = \begin{cases} \frac{\int_T \mu(t)h(t)d\lambda}{\int_T h^2(t)d\lambda} = \int_T \mu(t)h(t)d\lambda(t)/h_0 & \text{if } h \not\equiv 0, \\ 0 & \text{if } h \equiv 0. \end{cases} \tag{6}$$

This parameter μ_0 will be called *intrinsic riskless parameter.* Finally, for each $n \geq 1$, let $\mu_n = (\mu, \psi_n) = \int_T \mu(t)\psi_n(t)d\lambda$. If we view ψ_n as a portfolio, μ_n and s_n are respectively the mean $E(\psi_n)$ and cost $C(\psi_n)$ of ψ_n. Since $\mu \in L^2(T)$ from (2) above, we can also project it on the closed subspace spanned by the constant function 1 and $\mathcal{L}$, henceforth $(\mathcal{L} \vee \{1\})$. Let μ_s be the projection of μ on $(\mathcal{L} \vee \{1\})^{\perp}$. Also, let $(\mathcal{L} \vee \{h\})$ be the closed subspace spanned by $[h, \psi_1(\cdot), \psi_2(\cdot), \cdots]$. Since $(\mathcal{L} \vee \{1\})$ and $(\mathcal{L} \vee \{h\})$ are identical, we obtain

$$\mu = \mu_0 h + \sum_{n=1}^{\infty} \mu_n \psi_n + \mu_s. \tag{7}$$

Next, we present a formulation of an idea that is the conceptual driving force of the theory EAPT developed in [16].

Definition 2 *A market does not permit exact arbitrage opportunities if for any portfolio* p, $V(p) = C(p) = 0$ *implies* $E(p) = 0$.

In the introduction, we mention how this notion of exact arbitrage can be shown to be equivalent to an asset-pricing formula and thereby provide an analogue to Ross' theorem based on specific restrictions on the distribution of asset returns, the so-called strict factor structure. As the motivating questions in the introduction make clear, our concern here is not with this formula, and so we record this benchmark result in a form more amenable to the analysis developed in the sequel [15, Corollary 1].

Theorem 1 *A market does not permit exact arbitrage opportunities if and only if* $\mu_s = 0$.

In conclusion we present an existence result which will allow us (in Remarks 1 to 10 below) to manufacture variety of examples of specific financial markets that illustrate the various conditions isolated in Theorems 2 to 5 for the existence or non-existence of different kinds of portfolios that serve as important markers for the analysis; see Table 1 for an overview.

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[13] We shall use the symbol $\equiv$ to denote a definition, or, in the case of a measurable function, equality almost everywhere.

Proposition 1 *For all sequences* $\{s_n\}_{n=1}^{\infty}$ *and* $\{\mu_n\}_{n=1}^{\infty}$ *of real numbers, and a decreasing sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *of nonnegative numbers with* $\sum_{n=1}^{\infty} s_n^2 \le 1$, $\sum_{n=1}^{\infty} \mu_n^2 < \infty$, *and* $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$, *for each given distribution* ν *on* $\mathbb{R}$ *with zero mean and unit variance, for all real numbers* $\bar{\mu}_0$ *and* $a \ge 0$, *there exist (1)* x *and* e *in* $L^2(T \times \Omega)$, *(2) a sequence of orthonormal functions* $\{\psi_n\}_{n=1}^{\infty}$ *on* $(T, \mathcal{T}, \lambda)$, *and (3) a sequence of orthonormal centered random variables* $\{\varphi_n\}_{n=1}^{\infty}$ *on* $(\Omega, \mathcal{A}, P)$, *such that*

- *(i)* $\int_T \psi_n(t) d\lambda(t) = s_n$ *for all* $n \ge 1$, *and* $h = 1 - \sum_{n=1}^{\infty} s_n \psi_n$,
- *(ii) for* λ*-almost all* $t \in T$, e_t *has distribution* ν, *and for* $\lambda \otimes \lambda$*-almost all* $(s,t) \in T \times T$, e_s *and* e_t *are independent,*[14]
- *(iii) there exists* $\mu_s \perp (\mathcal{L} \vee \{1\})$ *with* $\int_T \mu_s^2(t) d\lambda(t) = a$, *and* $\mu = \bar{\mu}_0 h + \sum_{n=1}^{\infty} \mu_n \psi_n + \mu_s$,[15]
- *(iv)* $x(t,\omega) = \mu(t) + \sum_{n=1}^{\infty} \lambda_n \psi_n(t) \varphi_n(\omega) + e(t,\omega)$.

We relegate the proof of this result to the Appendix, and turn to a presentation of our substantive results.

4 Normalized Riskless Portfolios and Factor Portfolios

In this section we offer necessary and sufficient conditions for the existence of a normalized riskless portfolio, as well as for all of the factors $\varphi_n(\cdot)$ to be portfolio returns. In the first case, we have to show the existence of a portfolio which has cost one, and whose non-zero return is independent of the state of nature $\omega \in \Omega$. In the second case, we have to find portfolios whose random returns are precisely φ_n. We begin with a formal specification of the notion of a normalized riskless portfolio.

Definition 3 *A portfolio* r *is a normalized riskless portfolio if* $V(r) = 0$, $C(r) = 1$ *and* $E(r) \neq 0$.

The following theorem characterizes the existence of a normalized riskless portfolio. Unlike [7, Definition 1], our definition allows us to consider markets which offer exact arbitrage opportunities, and thereby present a complete characterization. Also unlike the characterization of the existence of a riskless portfolio in [7, Proposition 2], where a general parameter φ needs to be checked for every N, we have a very simple characterization here in terms of the intrinsic riskless portfolio and a parameter pertaining to it, and a parameter formalizing exact arbitrage opportunities.

Theorem 2 *There is a normalized riskless portfolio if and only if one of the following holds: the intrinsic riskless portfolio* h *is not null and either (i) the intrinsic riskless parameter* $\mu_0 \neq 0$*; or (ii)* $\mu_0 = 0$*, and the market permits exact arbitrage opportunities.*

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[14] A process satisfying these two conditions are called almost i.i.d. in [29].

[15] According to the definition of μ_0 in Equation (6), if $h \equiv 0$, then $\mu_0 = 0$; however, $\bar{\mu}_0$ may not be zero.

Proof: If $\mu_0 \neq 0$, let $r(t) = h(t)/h_0$. Then it is easy to check using Equations (3), (6) and (7) that $C(r) = 1$ and the random return $\mathcal{R}_r \equiv \mu_0$. Thus $V(r) = 0$, $E(r) = \mu_0 \neq 0$, and hence r is a normalized riskless portfolio. Next, we consider the second case when $\mu_0 = 0$, and both h and μ_s are not null functions, the latter because of Theorem 1. Let $r'(t) = h(t)/h_0 + \mu_s(t)$. Then by Equations (3), (6) and (7) again, we obtain $C(r') = 1$ and $\mathcal{R}_{r'} \equiv \int_T \mu_s^2(t)d\lambda(t)$. Since μ_s is not null, $E(r') > 0$, and hence r' is a normalized riskless portfolio.

Now, we assume that the market has a normalized riskless portfolio r''. On projecting $r''(t)$ to the subspace $(\mathcal{L} \vee \{h\})$, we obtain $r''(t) = r_0 h(t) + \sum_{n=1}^{\infty} r_n \psi_n(t) + r_s(t)$, where $r_s(t) \perp (\mathcal{L} \vee \{h\})$. Then by Equation (3), $\mathcal{R}_{r''} = \int_T r''(t)\mu(t)d\lambda(t) + \sum_{n=1}^{\infty} r_n \lambda_n \varphi_n(\omega)$. Since $V(r'') = 0$, we obtain $r_n = 0$ for all $n \geq 1$. This fact, together with Equation (7), implies that

$$\begin{aligned}
\mathcal{R}_{r''} &= \int_T r''(t)\mu(t)d\lambda(t) = r_0 \int_T h(t)\mu(t)d\lambda(t) + \int_T r_s(t)\mu(t)d\lambda(t) \\
&= r_0\mu_0 \int_T h^2(t)d\lambda(t) + \int_T r_s(t)\mu_s(t)d\lambda(t) \\
&= r_0\mu_0 h_0 + \int_T r_s(t)\mu_s(t)d\lambda(t)
\end{aligned}$$

Furthermore,

$$C(r'') = \int_T r''(t)d\lambda(t) = r_0 h_0.$$

Since we require that $C(r'') = 1$, we must have h nonnull. On the other hand, $E(r'')$ is assumed to be nonnull. Hence we cannot have both $\mu_0 = 0$ and $\mu_s \equiv 0$. On combining these conditions together, we obtain either condition (i) or condition (ii). ■

Proposition 1 allows one to construct various examples to show the existence or non-existence of normalized riskless portfolios in the market. The following remarks show that every condition in Theorem 2 is non-vacuous.[16]

Remark 1 *By Theorem 2, the following two markets have no normalized riskless portfolios. (i) Take* $s_1 = 1$ *and* $s_n = 0$ *for all* $n \geq 2$ *to obtain a market with* $h \equiv 0$. *Note that we can choose* a *to be zero or nonzero so that* $\mu_s \equiv 0$ *or* $\mu_s \not\equiv 0$. *(ii) Take* $a = 0$, $\bar{\mu}_0 = 0$, $s_1 = 1/2$, $s_n = 0$ *for all* $n \geq 2$ *to obtain a market with* $h \not\equiv 0$, $\mu_0 = 0$ *and* $\mu_s \equiv 0$.

Remark 2 *By Theorem 2, the following three markets have normalized riskless portfolios. (i) Take* $a = 0$, $\bar{\mu}_0 = 1$, $s_1 = 1/2$, $s_n = 0$ *for all* $n \geq 2$ *to obtain a market with* $h \not\equiv 0$, $\mu_0 = 1$ *and* $\mu_s \equiv 0$. *(ii) Take* $a = 1$, $\bar{\mu}_0 = 0$, $s_1 = 1/2$, $s_n = 0$ *for all* $n \geq 2$ *to obtain a market with* $h \not\equiv 0$, $\mu_0 = 0$ *and* $\mu_s \not\equiv 0$. *(iii) Take* $a = 1$, $\bar{\mu}_0 = 1$, $s_1 = 1/2$, $s_n = 0$ *for all* $n \geq 2$ *to obtain a market with* $h \not\equiv 0$, $\mu_0 = 1$ *and* $\mu_s \not\equiv 0$.

In the following corollary, we do assume that the market does not permit exact arbitrage opportunities. Since $h \equiv 0$ implies that $\mu_0 = 0$, the result singles out the importance of the intrinsic riskless parameter μ_0 for the existence of a normalized riskless portfolio. In comparison with [7, Proposition 2], we can give an explicit formula for all the possible portfolios which make up the normalized riskless portfolio.

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[16] We shall not specify that in Remarks 1 to 10 below, the choice of parameters has Proposition 1 as their relevant context.

Corollary 1 *Assume that the market does not permit exact arbitrage opportunities. Then there is a normalized riskless portfolio if and only if* $\mu_0 \neq 0$. *In this case,* r *is a normalized riskless portfolio if and only if* $r = h/h_0 + r_s$, *where* r_s *is a portfolio such that* $r_s \perp h$ *and* $r_s \perp \psi_n$ *for all* $n \geq 1$. *Moreover, the return of a normalized riskless portfolio must be* μ_0.

Proof: The first result is clear from Theorem 1. Then we assume $\mu_0 \neq 0$. Let r be a portfolio with the expansion $r(t) = r_0 h(t) + \sum_{n=1}^{\infty} r_n \psi_n(t) + r_s(t)$, as the portfolio r'' in the proof of Theorem 2. If r is a normalized riskless portfolio, then $C(r) = r_0 h_0 = 1$ and $r_n = 0$ for all $n \geq 1$, and hence $r = h/h_0 + r_s$, where $r_s \perp h$ and $r_s \perp \psi_n$ for all $n \geq 1$. The rest is clear. ■

When the market does not permit exact arbitrage opportunities, let us call a portfolio q a dummy portfolio if $q \perp h$ and $q \perp \psi_n$ for all $n \geq 1$. Since $L^2(T)$ is non-separable, there are plenty of dummy portfolios. So Corollary 1 simply says that r is a normalized riskless portfolio if and only if r is the sum of h/h_0 plus a dummy portfolio. The following lemma gives some simple sufficient conditions for a portfolio to be dummy.

Lemma 1 *Assume that the market does not permit exact arbitrage opportunities. Let* p *be a portfolio. Then*

- *(i)* p *is a dummy portfolio if and only if* $E(p) = C(p) = V(p) = 0$;
- *(ii) if either* $\mu_0 \neq 0$ *or* $h \equiv 0$, *then any portfolio* p, *with* $\mathcal{R}_p \equiv 0$ *must be a dummy portfolio.*

Proof: The necessity part of (i) is clear. To prove the sufficiency part of (i), assume that $E(p) = C(p) = V(p) = 0$. By Equation (3), $p \perp \psi_n$ for all $n \geq 1$. Since $C(p) = 0$, $p \perp 1$, and hence p is dummy.

For (ii), project p to $(\mathcal{L} \vee \{h\})$ to obtain $p = p_0 h + \sum_{n=1}^{\infty} p_n \psi_n + p_s$. By Equation (3), $\mathcal{R}_p = \int_T p(t)\mu(t)d\lambda(t) + \sum_{n=1}^{\infty} p_n \lambda_n \varphi_n$. Hence the assumption $\mathcal{R}_p \equiv 0$ implies that $(p, \mu) = \int_T p(t)\mu(t)d\lambda(t) = 0$, and $p_n = 0$ for all $n \geq 1$. Since $\mu = \mu_0 h + \sum_{n=1}^{\infty} \mu_n \psi_n$ in the absence of exact arbitrage opportunities, $(p, \mu) = \mu_0 p_0 h = 0$, and hence $p \perp h$ under the hypothesis that either $\mu_0 \neq 0$ or $h \equiv 0$, and thus p is dummy. ■

Next, we characterize those markets which allow all factors to be portfolio returns. In the following theorem, as in Theorem 1, we do not assume the absence of exact arbitrage opportunities in the market.

Theorem 3 *All the factors* φ_n *are portfolio returns if and only if one of the following holds: (i)* $\mu_0 \neq 0$; *(ii)* $\mu(t) = 0$ *for* λ*-almost all* $t \in T$; *or (iii) the market permits exact arbitrage opportunities.*

Proof: First assume that all the factors φ_n are portfolio returns. Then there exists portfolios $q^n(t)$ such that $\mathcal{R}_{q^n} = \varphi_n$ for all $n \geq 1$. By projecting $q^n(t)$ to $(\mathcal{L} \vee \{h\})$, we obtain

$$q^n(t) = q_0^n h(t) + \sum_{m=1}^{\infty} q_m^n \psi_m(t) + q_s^n(t),$$

where $q_s^n(t) \perp (\mathcal{L} \vee \{h\})$. Then by Equation (3),

$$\mathcal{R}_{q^n}(\omega) = \int_T q^n(t)\mu(t)d\lambda(t) + \sum_{m=1}^{\infty} \lambda_m q_m^n \varphi_m(\omega).$$

For each $n \geq 1$, by the fact that $\mathcal{R}_{q^n} = \varphi_n$, we obtain $\int_T q^n(t)\mu(t)d\lambda(t) = 0$ and $q_n^n = (1/\lambda_n)$, and $q_m^n = 0$ if $1 \leq m \neq n < \infty$. By Equation (7), we can obtain

$$\int_T q^n(t)\mu(t)d\lambda(t) = q_0^n \mu_0 h_0 + \sum_{m=1}^{\infty} q_m^n \mu_m + \int_T q_s^n(t)\mu_s(t)d\lambda(t) = 0,$$

and hence $q_0^n \mu_0 h_0 + (\mu_n/\lambda_n) + \int_T q_s^n(t)\mu_s(t)d\lambda(t) = 0$ for all $n \geq 1$.

If (i), (ii) and (iii) in the statement of Theorem 3 all fail, then $\mu_0 = 0$, $\mu \not\equiv 0$ and $\mu_s \equiv 0$. Then, $\mu_0 = 0$ and $\mu_s \equiv 0$ imply that $q_0^n \mu_0 h_0 + (\mu_n/\lambda_n) + \int_T q_s^n(t)\mu_s(t)d\lambda(t) = \mu_n/\lambda_n = 0$ for all $n \geq 1$. Thus $\mu_n = 0$ for all $n \geq 1$, and hence $\mu \equiv 0$, which contradicts with $\mu \not\equiv 0$. Therefore if all the factors φ_n are portfolio returns, then one of (i), (ii) and (iii) must hold.

Next assume that $\mu_0 \neq 0$. For all $n \geq 1$, let

$$p_n(t) = -\frac{\mu_n}{\lambda_n \mu_0 h_0} h(t) + \frac{1}{\lambda_n} \psi_n(t).$$

Then, by Equations (3), (6) and (7), we obtain

$$\begin{aligned} \mathcal{R}_{p_n} &= \int_T p_n(t)\mu(t)d\lambda(t)) + \varphi_n \\ &= -\frac{\mu_n}{\lambda_n \mu_0 h_0} \cdot \mu_0 \int_T h^2(t)d\lambda(t) + \frac{\mu_n}{\lambda_n} + \varphi_n \\ &= \varphi_n. \end{aligned}$$

Thus, each φ_n is a portfolio return with $E(p_n) = \int_\Omega \varphi_n(\omega)dP(\omega) = 0$, $V(p_n) = 1$, and

$$\begin{aligned} C(p_n) &= \int_T p_n(t)d\lambda(t) = -\frac{\mu_n}{\lambda_n \mu_0 h_0} \cdot \int_T h(t)d\lambda(t) + \frac{1}{\lambda_n} s_n \\ &= \frac{1}{\lambda_n}\left[s_n - \frac{\mu_n}{\mu_0}\right]. \end{aligned}$$

Now if (ii) holds, one can simply take $q^n(t) = (\psi_n(t)/\lambda_n)$ to obtain that $\mathcal{R}_{q^n} = \varphi_n$. Finally, if (iii) holds, let

$$q^n(t) = \frac{\psi_n(t)}{\lambda_n} - \frac{\mu_n}{\lambda_n \int_T \mu_s^2(t)d\lambda(t)} \mu_s(t).$$

The rest is easy to check. ∎

The following remarks show that every case considered in Theorem 3 is possible.

Remark 3 *Take $a = 0$, $\bar{\mu}_0 = 0$, $\mu_1 = 1$, $\mu_n = 0$ for all $n \geq 2$ to obtain a market with $\mu \not\equiv 0$, $\mu_0 = 0$ and $\mu_s \equiv 0$. By proper choices of s_n, as in Remark 1 above, we can further specify the market to obtain $h \equiv 0$ or $h \not\equiv 0$. In either case, Theorem 3 shows that not all factors are portfolio returns in these markets.*

Remark 4 *By Theorem 3, all factors are portfolio returns in the following three markets. (i) The market in Remark 2(i) has* $\mu_0 \neq 0$*. (ii) The market in Remark 2(ii) has* $\mu_s \not\equiv 0$*. (iii) Take* $\bar{\mu}_0 = 0$*,* $\mu_n = 0$ *for all* $n \geq 1$ *to obtain a market with* $\mu \equiv 0$*.*

The following corollary is a simplification of Theorem 3 in a setting where there are no exact arbitrage opportunities.

Corollary 2 *Under the assumption that the market does not permit exact arbitrage opportunities, all the factors are portfolio returns if and only if either* $\mu_0 \neq 0$ *or* $\mu(t) = 0$ *for* λ*-almost all* $t \in T$*.*

If $\mu_0 \neq 0$ and the market does not permit exact arbitrage opportunities, there is a normalized riskless portfolio and all the factors are portfolio returns. In fact, for any random variable of the form $a_0 + \sum_{n=1}^{\infty} a_n \varphi_n(\omega)$ with $\sum_{n=1}^{\infty} a_n^2/\lambda_n^2 < \infty$, there is a portfolio with this random return.

5 Mean and Cost Portfolios

In the previous section, we looked for portfolios whose associated random returns in the space $L^2(\Omega)$ have a given form. In this section, we look for portfolios whose random returns furnish the mean and cost of the asset return x_t for λ-almost all $t \in T$, which is to say, the continuous linear functionals on $L^2(\Omega)$ defined by the random returns of these portfolios give the values μ_t and 1 to the elements x_t, for λ-almost all $t \in T$. Formally, a portfolio m is said to be a mean portfolio if the inner product $(\mathcal{R}_m, x_t) = \mu_t$ for λ-almost all $t \in T$. A portfolio c is said to be a cost portfolio if the inner product $(\mathcal{R}_c, x_t) = 1$ for λ-almost all $t \in T$. We develop necessary and sufficient conditions for the existence of such portfolios. Mean and cost portfolios will provide the basis for constructing various kinds of mean-variance efficient portfolios in the next section.

The following simple lemma shows that the mean and cost portfolios, as defined above, not only give the right means and costs for individual assets but also for all portfolios. This is consistent with the usual practice in the literature; see, for example, [26].

Lemma 2 *If* m *is a mean portfolio, then for any portfolio* p*, the expected return* $E(p)$ *of the portfolio* p *is* $(\mathcal{R}_m, \mathcal{R}_p) = \int_\Omega \mathcal{R}_m \mathcal{R}_p dP(\omega)$*. Analogously, if* c *is a cost portfolio, then for any portfolio* p*, the cost* $C(p)$ *of the portfolio* p *is* $(\mathcal{R}_c, \mathcal{R}_p) = \int_\Omega \mathcal{R}_c \mathcal{R}_p dP(\omega)$*.*

Proof: Since $\mathcal{R}_p = \int_T p(t) x(t,\omega) d\lambda(t)$, $(\mathcal{R}_m, \mathcal{R}_p) = \int_\Omega \mathcal{R}_m(\omega) \int_T p(t) x(t,\omega) d\lambda(t) dP(\omega)$. Hence, by the definition of a mean portfolio,

$$(\mathcal{R}_m, \mathcal{R}_p) = \int_T p(t) \int_\Omega \mathcal{R}_m(\omega) x(t,\omega) dP(\omega) d\lambda(t) = \int_T p(t) \mu(t) d\lambda(t) = E(p),$$

and we are done. The proof for the case of cost portfolios is the same. ■

We shall now characterize the existence of a mean portfolio. The explicit formula for such a portfolio is also given with the proof.

Theorem 4 *If either the intrinsic riskless parameter* $\mu_0 \neq 0$ *or the market has exact arbitrage opportunities, i.e.* μ_s *is not null, then a mean portfolio exists. On the other hand, if* $\mu_0 = 0$ *and* $\mu_s \equiv 0$*, then there is a mean portfolio if and only if* $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^4) < \infty$.

Proof: As in the proof of Theorem 2 , if $\mu_0 \neq 0$ or μ_s is not null, we can respectively define a portfolio m by $m(t) = h(t)/\mu_0 h_0$ or by $m(t) = (\mu_s(t)/\int_T \mu_s^2(t)d\lambda(t)$. It is clear that $\mathcal{R}_m \equiv 1$, and hence $E(p) = (\mathcal{R}_m, \mathcal{R}_p)$ for any portfolio p. Therefore m is a mean portfolio.

It remains to consider the case when $\mu_0 = 0$ and $\mu_s \equiv 0$. Let m be a mean portfolio, i.e., $(\mathcal{R}_m, x_t) = \mu_t$ for λ-almost all $t \in T$. Let $m(t) = m_0 h(t) + \sum m_n \psi_n(t) + m_s(t)$, where $m_s \perp h$ and $m_s \perp \psi_n$ for all $n \geq 1$. Note that $\mathcal{R}_m(\omega) = \int_T m(t)\mu(t)d\lambda(t) + \sum_{n=1}^{\infty} m_n \lambda_n \varphi_n(\omega)$ and $E(m) = \int_T m(t)\mu(t)d\lambda(t)$. Then for λ-almost all $t \in T$, $(\mathcal{R}_m, x_t) = \mu(t)E(m) + \sum_{n=1}^{\infty} m_n \lambda_n^2 \psi_n(t) = \mu(t)$. By the fact that $\mu(t) = \sum_{n=1}^{\infty} \mu_n \psi_n(t)$, we obtain that

$$\left(\sum_{n=1}^{\infty} \mu_n \psi_n(t)\right)(E(m) - 1) + \sum_{n=1}^{\infty} m_n \lambda_n^2 \psi_n(t) = 0.$$

Thus, for λ-almost all $t \in T$, $\sum_{n=1}^{\infty}\left(\mu_n(E(m) - 1) + m_n\lambda_n^2\right)\psi_n(t) = 0$. Hence it follows from the orthogonality of the ψ_n that $\mu_n(E(m) - 1) + m_n\lambda_n^2 = 0$ for each $n \geq 1$. This means that $m_n = (\mu_n(1 - E(m)))/\lambda_n^2$. If $E(m) = 1$, then $m_n = 0$; since $E(m) = \sum_{n=1}^{\infty} m_n\mu_n$, we also obtain $E(m) = 0$. This is a contradiction. Therefore $E(m) \neq 1$. By the fact that $\sum_{n=1}^{\infty} m_n^2 < \infty$, we obtain that $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^4 < \infty$.

On the other hand, if $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^4 < \infty$, then we can simply define a portfolio m by letting

$$m(t) = \sum_{n=1}^{\infty} m_n \psi_n(t), \; m_n = \left(1 + \sum_{n=1}^{\infty} \frac{\mu_n^2}{\lambda_n^2}\right)^{-1} \frac{\mu_n}{\lambda_n^2}, \; n \geq 1.$$

Note that the convergence of the series $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^4$ implies the convergence of $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2$, since $\lim_{n\to\infty} \lambda_n = 0$. Thus the above portfolio m is well-defined. By the previous computation, it is clear that m is a mean portfolio with $E(m) = (\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2)/(1 + \sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2)$, and we are done. ■

The following remark shows the existence of a market in which the conditions of Theorem 4 fail, and thus it has no mean portfolio.

Remark 5 *Take* $a = 0$, $\bar{\mu}_0 = 0$, $\mu_n = 1/n$ *and* $\lambda_n = 1/n$ *for* $n \geq 1$ *to obtain a market with* $\mu_0 = 0$, $\mu_s \equiv 0$, *and* $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^4) = \sum_{n=1}^{\infty} n^2 = \infty$. *By proper choices of* s_n*, we can further specify the market to obtain* $h \equiv 0$ *or* $h \not\equiv 0$*. Theorem 4 shows that there is no mean portfolio in this market.*

The next remark shows that there are markets satisfying each of the three conditions in Theorem 4 that guarantee the existence of a mean portfolio.

Remark 6 *Theorem 4 shows that there is a mean portfolio in the following markets. Note that we can further specify the market to obtain* $h \equiv 0$ *or* $h \not\equiv 0$ *by proper choices of* s_n.
(i) Remark 2 contains examples of markets with $\mu_0 \neq 0$ *or* $\mu_s \not\equiv 0$*. (ii) Take* $a = 0$, $\bar{\mu}_0 = 0$, *and* $\mu_n = \lambda_n^3$ *for* $n \geq 1$ *to obtain a market with* $\mu_0 = 0$, $\mu_s \equiv 0$, *and* $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^4) = \sum_{n=1}^{\infty} \lambda_n^2 < \infty$.

The next theorem characterizes the existence of a cost portfolio. As in the case of Theorem 4, explicit formulae are also included in the proof. Parts (i) and (ii) of the following theorem are available in [16, Proposition 3], which are included here for the sake of completeness.

Theorem 5 *(i) If the intrinsic riskless parameter $\mu_0 \neq 0$, then a cost portfolio exists if and only if the market does not permit exact arbitrage opportunities, i.e., $\mu_s \equiv 0$, and $\sum_{n=1}^{\infty} (\mu_0 s_n - \mu_n)^2 / \lambda_n^4 < \infty$.*

(ii) If $\mu_0 = 0$ and the intrinsic riskless portfolio h is not null, then there is no cost portfolio.

(iii) If $h \equiv 0$ and $\mu_s \not\equiv 0$, then a cost portfolio exists if and only if $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$.

(iv) If $h \equiv 0$, $\mu_s \equiv 0$, and $\sum_{n=1}^{\infty}(s_n^2 + \mu_n^2)/\lambda_n^2 < \infty$, then there is a cost portfolio if and only if $\sum_{n=1}^{\infty}(s_n - b\mu_n)^2/\lambda_n^4 < \infty$, where $b = (\sum_{n=1}^{\infty}(\mu_n s_n/\lambda_n^2))/(1 + \sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2))$.

Proof: Let $c(t) = c_0 h(t) + \sum c_n \psi_n(t) + c_s(t)$ be a cost portfolio, i.e., $(\mathcal{R}_c, x_t) = 1$ for λ-almost all $t \in T$, where c_s is orthogonal to h and all the ψ_n. By Equation (3),

$$\mathcal{R}_c(\omega) = \int_T c(t)\mu(t)d\lambda(t) + \sum_{n=1}^{\infty} c_n \lambda_n \varphi_n(\omega)$$

and $E(c) = \int_T c(t)\mu(t)d\lambda(t)$. Then λ-almost all $t \in T$,

$$\begin{aligned}
(\mathcal{R}_c, x_t) &= E(c)\mu(t) + \sum_{n=1}^{\infty} c_n \lambda_n^2 \psi_n(t) \\
&= E(c)\mu_0 h(t) + \sum_{n=1}^{\infty} \left(E(c)\mu_n + c_n\lambda_n^2\right) \psi_n(t) + E(c)\mu_s(t) \\
&= 1 = h(t) + \sum_{n=1}^{\infty} s_n \psi_n(t).
\end{aligned}$$

By the fact that μ_s, h, and all the ψ_n are mutually orthogonal, we can obtain from the above formula that $E(c)\mu_0 h = h$, $E(c)\mu_n + c_n\lambda_n^2 = s_n$, and $E(c)\mu_s \equiv 0$.

For the proof of (i), note that $\mu_0 \neq 0$ implies that h is not null. Consider a cost portfolio c as above. By the identity $E(c)\mu_0 h = h$, we have $E(c) = 1/\mu_0$. The identity $E(c)\mu_s \equiv 0$ implies $\mu_s \equiv 0$. Moreover, for $n \geq 1$, $c_n = (\mu_0 s_n - \mu_n)/(\mu_0 \lambda_n^2)$. Since c is a portfolio, we have $\sum_{n=1}^{\infty} c_n^2 < \infty$, and hence $\sum_{n=1}^{\infty}((\mu_0 s_n - \mu_n)/\lambda_n^2)^2 < \infty$. On the other hand, if $\mu_0 \neq 0$, $\mu_s = 0$, and $\sum_{n=1}^{\infty}((\mu_0 s_n - \mu_n)/\lambda_n^2)^2 < \infty$, then we can define a portfolio c' by letting

$$c'(t) = c'_0 h(t) + \sum_{n=1}^{\infty} c'_n \psi_n(t),$$

and where

$$c'_0 = \frac{(1/\mu_0) - \sum_{n=1}^{\infty} c'_n \mu_n}{\mu_0 h_0}, \text{ and } c'_n = \frac{\mu_0 s_n - \mu_n}{\mu_0 \lambda_n^2} \quad \text{for all } n \geq 1.$$

It is easy to check that $E(c') = 1/\mu_0$. By the computation in the previous paragraph, it is clear that c' is a cost portfolio.

For the proof of (ii), consider $\mu_0 = 0$ and h is not null. Suppose c is a cost portfolio. Then as shown in the first paragraph of this proof, $E(c)\mu_0 h = h$. This implies that $h \equiv 0$, which contradicts the hypothesis. Therefore there is no cost portfolio in this case. In fact, in this case, the portfolio h has a positive cost $C(h) = \int_T h d\lambda(t) = \int_T h^2 d\lambda(t)$ but with random return $\mathcal{R}_h \equiv 0$. Thus $C(h)$ is not equal to the inner product of $\mathcal{R}_h$ with the random return of any other portfolio.

For (iii), we move to the case $h \equiv 0$ and $\mu_s \not\equiv 0$. If there is a cost portfolio c, then it follows from the identity $E(c)\mu_s \equiv 0$ in the first paragraph that $E(c) = 0$. Hence $c_n = s_n/\lambda_n^2$. By the fact that $\sum_{n=1}^{\infty} c_n^2 < \infty$, we have $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$.

On the other hand, if $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$, then we can define a portfolio c' by letting

$$c'(t) = \sum_{n=1}^{\infty} \left(\frac{s_n}{\lambda_n^2}\right) \psi_n(t) - \left(\sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2 \int_T \mu_s^2 d\lambda(t)}\right) \mu_s(t).$$

It is easy to check that $E(c') = 0$. By the computation in the first paragraph, it is clear that c' is a cost portfolio.

Finally, for (iv), we consider the case when $h \equiv 0$, $\mu_s \equiv 0$, and $\sum_{n=1}^{\infty}(s_n^2 + \mu_n^2)/\lambda_n^2 < \infty$. By the Cauchy-Schwartz inequality, it is clear that the series $\sum_{n=1}^{\infty} \mu_n s_n/\lambda_n^2$ is convergent. Thus b is well defined.

Let c be a cost portfolio as in the first paragraph. Then we obtain $c_n = (s_n - E(c)\mu_n)/\lambda_n^2$ for all $n \geq 1$. Now

$$E(c) = \sum_{n=1}^{\infty} c_n \mu_n = \sum_{n=1}^{\infty} \mu_n (s_n - E(c)\mu_n)/\lambda_n^2 = \sum_{n=1}^{\infty} (\mu_n s_n)/\lambda_n^2 - E(c) \sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2,$$

which implies that $E(c) = b$. By the fact that $\sum_{n=1}^{\infty} c_n^2 < \infty$, we obtain that $\sum_{n=1}^{\infty}(s_n - b\mu_n)^2/\lambda_n^4 < \infty$.

On the other hand, if $\sum_{n=1}^{\infty}(s_n - b\mu_n)^2/\lambda_n^4 < \infty$, then define a portfolio c' by letting

$$c'(t) = \sum_{n=1}^{\infty} c'_n \psi_n(t), \; c'_n = \frac{s_n - b\mu_n}{\lambda_n^2}, \; n \geq 1.$$

It is easy to check that $E(c') = b$. By the computation in the previous paragraph, it can be checked that c' is indeed a cost portfolio. ■

For Case (i) of Theorem 5, Theorem 4 says that a mean portfolio always exists. The following remark shows that each condition in Theorem 5(i) is realized in some market. An example similar to that in Part (ii) of the following remark is furnished as Example 7 in [16].

Remark 7 *Theorem 5(i) shows that there is a cost portfolio in the third market but not in the first two markets.*
(i) Take $a = 1$, $\bar{\mu}_0 = 1$, $s_1 = 1/2$, $s_n = 0$ *for all* $n \geq 2$ *to obtain a market with* $h \not\equiv 0$, $\mu_0 = 1$, *and* $\mu_s \not\equiv 0$. *(ii) Take* $a = 0$, $\bar{\mu}_0 = 1$, $s_1 = 1/2$, $s_n = 0$ *for all* $n \geq 2$, *and* $\mu_n = \lambda_n = 1/n$ *for* $n \geq 1$ *to obtain a market with* $\mu_0 \neq 0$, $\mu_s \equiv 0$, *and* $\sum_{n=1}^{\infty} (\mu_0 s_n - \mu_n)^2/\lambda_n^4 = \infty$. *(iii) Take* $a = 0$, $\bar{\mu}_0 = 1$, $s_1 = \mu_1 = 1/2$, *and* $s_n = \mu_n = 0$ *for all* $n \geq 2$ *to obtain a market with* $\mu_0 \neq 0$, $\mu_s \equiv 0$, *and* $\sum_{n=1}^{\infty} (\mu_0 s_n - \mu_n)^2/\lambda_n^4 < \infty$.

For Case (ii) of Theorem 5, there is definitely no cost portfolio. However, the following remark shows that a mean portfolio may or may not exist in this case. Example 6 in [16] shares some of the spirit of the example in Part (ii) of the remark.

Remark 8 *Theorems 4 and 5(ii) show that there is a mean portfolio but no cost portfolio in the first market but neither kind of portfolio in the second market.*
(i) Take $a = 1$*,* $\bar{\mu}_0 = 0$*,* $s_1 = 1/2$*,* $s_n = 0$ *for all* $n \geq 2$ *to obtain a market with* $h \not\equiv 0$*,* $\mu_0 = 0$*, and* $\mu_s \not\equiv 0$*. (ii) Take* $a = 0$*,* $\bar{\mu}_0 = 0$*,* $s_1 = 1/2$*,* $s_n = 0$ *for all* $n \geq 2$*, and* $\mu_n = \lambda_n = 1/n$ *to obtain a market with* $h \not\equiv 0$*,* $\mu_0 = 0$*,* $\mu_s \equiv 0$*, and* $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^4) = \infty$*.*

For Case (iii) of Theorem 5, Theorem 4 also guarantees the existence of a mean portfolio. However, the following remark shows that a cost portfolio may or may not exist in this case. An example similar to that in Part (i) of the remark appeared as Example 5 in [16].

Remark 9 *Theorem 5(iii) shows that there is a cost portfolio in the first market but not in the second. Note that the first market allows the possibility of gains from exact arbitrage and yet it has both a mean and a cost portfolio.*
(i) Take $a = 1$*,* $s_1 = s_2 = 1/\sqrt{2}$*,* $s_n = 0$ *for all* $n \geq 3$ *to obtain a market with* $h \equiv 0$*,* $\mu_s \not\equiv 0$*, and* $\sum_{n=1}^{\infty}(s_n^2/\lambda_n^4) < \infty$*. (ii) Take* $a = 1$*,* $s_n = \lambda_n = 2^{-n/2}$ *for all* $n \geq 1$ *to obtain a market with* $h \equiv 0$*,* $\mu_s \not\equiv 0$*, and* $\sum_{n=1}^{\infty}(s_n^2/\lambda_n^4) = \infty$*.*

The last remark of this section concerns Case (iv) of Theorem 5. We have seen from previous remarks that a mean portfolio may exist but with or without a cost portfolio, or both do not exist. These three possibilities can also happen in Case (iv) of Theorem 5 as shown in Parts (i) - (iii) of the following remark. However, the example in Part (iv) of the remark illustrates a fourth possibility; namely, a market in which a cost portfolio may exist without a mean portfolio.

Remark 10 *Theorems 4 and 5(iv) show that the first market has both a mean and a cost portfolio, the second has neither, the third has only a mean portfolio, and the fourth, only a cost portfolio.*

(i) Take $a = 0$*,* $s_1 = s_2 = \mu_1 = \mu_2 = 1/\sqrt{2}$*, and* $s_n = \mu_n = 0$ *for all* $n \geq 3$ *to obtain a market with* $h \equiv 0$*,* $\mu_s \equiv 0$*, and for which all the relevant series in Theorems 4 and 5(iv) are convergent.*

(ii) Take $a = 0$*,* $s_n = 2^{-n/2}$*,* $\mu_n = 2^{-3n/5}$ *and* $\lambda_n = 2^{-n/3}$ *for* $n \geq 1$ *to obtain a market with* $h \equiv 0$*,* $\mu_s \equiv 0$*, and* $\sum_{n=1}^{\infty}(s_n^2/\lambda_n^2) = \sum_{n=1}^{\infty} 2^{-n/3} < \infty$*,* $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2) = \sum_{n=1}^{\infty} 2^{-8n/15} < \infty$*. It can be checked that* $\sum_{n=1}^{\infty}(s_n^2/\lambda_n^4) = \sum_{n=1}^{\infty} 2^{n/3} = \infty$*,* $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^4) = \sum_{n=1}^{\infty} 2^{2n/15} = \infty$*, and* $\sum_{n=1}^{\infty}(s_n - b\mu_n)^2/\lambda_n^4 = \sum_{n=1}^{\infty}(1 - b2^{-n/10})^2\, 2^{n/3} = \infty$*.*

(iii) Take $a = 0$*,* $s_n = 2^{-n/2}$*,* $\mu_n = 2^{-n}$ *and* $\lambda_n = 2^{-n/3}$ *for* $n \geq 1$ *to obtain a market with* $h \equiv 0$*,* $\mu_s \equiv 0$*, and* $\sum_{n=1}^{\infty}(s_n^2/\lambda_n^2) = \sum_{n=1}^{\infty} 2^{-n/3} < \infty$*,* $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2) = \sum_{n=1}^{\infty} 2^{-4n/3} < \infty$*. It can be checked that* $\sum_{n=1}^{\infty}(s_n^2/\lambda_n^4) = \sum_{n=1}^{\infty} 2^{n/3} = \infty$*,* $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^4) = \sum_{n=1}^{\infty} 2^{-2n/3} < \infty$*, and* $\sum_{n=1}^{\infty}(s_n - b\mu_n)^2/\lambda_n^4 = \sum_{n=1}^{\infty}(1 - b2^{-n/2})^2\, 2^{n/3} = \infty$*.*

(iv) *Let* $d = \sqrt{\frac{(2^{-5/6} - 2^{-4/3})}{(1-2^{-5/6})(1-2^{-4/3})}}$. *Take* $a = 0$, $s_n = 2^{-n/2}$, $\mu_n = (2^{-n/2} - 2^{-n})/d$ *and* $\lambda_n = 2^{-n/3}$ *for* $n \geq 1$ *to obtain a market with* $h \equiv 0$, $\mu_s \equiv 0$, *and* $\sum_{n=1}^{\infty}(s_n^2/\lambda_n^2) = \sum_{n=1}^{\infty} 2^{-n/3} < \infty$, $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2) = \sum_{n=1}^{\infty} 2^{-n/3}(1 - 2^{n/2})/d^2 < \infty$. *It can be checked that* $b = (\sum_{n=1}^{\infty}(\mu_n s_n/\lambda_n^2))/(1 + \sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2)) = d$, $\sum_{n=1}^{\infty}(s_n^2/\lambda_n^4) = \sum_{n=1}^{\infty} 2^{n/3} = \infty$, $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^4 = \sum_{n=1}^{\infty}(1-2^{-n/2})^2\, 2^{n/3}/d^2 = \infty$ *and* $\sum_{n=1}^{\infty}(s_n - b\mu_n)^2/\lambda_n^4 = \sum_{n=1}^{\infty} 2^{-2n/3} < \infty$.

Next, we turn to the question of uniqueness of the mean and cost portfolio returns. We first prove a simple proposition.

Proposition 2 *Let* p *be any portfolio. If the inner product* $(\mathcal{R}_p, x_t) = 0$ *for* λ*-almost all* $t \in T$, *then* $\mathcal{R}_p \equiv 0$.

Proof: It is easy to see that

$$(\mathcal{R}_p, \mathcal{R}_p) = \int_{\Omega} \mathcal{R}_p(\omega) \int_T p(t) x(t,\omega) d\lambda(t) dP(\omega) = \int_T p(t) \int_{\Omega} \mathcal{R}_p(\omega) x_t(\omega) dP(\omega) d\lambda(t) = 0,$$

and hence $\mathcal{R}_p \equiv 0$. ■

For any mean portfolios m_1, m_2 and any cost portfolios c_1, c_2, it is obvious that for λ-almost all $t \in T$, the inner products $(\mathcal{R}_{m_1 - m_2}, x_t)$ and $(\mathcal{R}_{c_1 - c_2}, x_t)$ are both equal to zero. The above proposition implies that $\mathcal{R}_{m_1} = \mathcal{R}_{m_2}$ and $\mathcal{R}_{c_1} = \mathcal{R}_{c_2}$.

6 Mean-Variance Efficient Portfolios

In this section we turn to mean-variance efficient portfolios. We also identify all those benchmark portfolios that can be used to measure the risk premium of any asset through a pricing formula based on a single portfolio proxying essential risk, the so-called beta pricing model of asset returns, see [16, Section 2.4]. In addition, we furnish explicit formulae for the measurement of the maximal trade-off between return and risk, the so-called maximal Sharpe measure, for all of the relevant situations. After presenting some basic definitions and results in Subsection 6.1, we consider in Subsection 6.2 the case where the mean and cost portfolio returns span a two-dimensional space, and in Subsection 6.4, a one-dimensional space. Subsection 6.3 covers the case where a cost portfolio does not exist. Each of the last three subsections describes in detail the mean-variance efficient and benchmark portfolios as well as the relevant maximal Sharpe measures. We shall assume throughout this section that the market does not permit gains from exact arbitrage opportunities; namely, that $\mu_s \equiv 0$.

6.1 Basic definitions and results

We begin with a formal definition of mean-variance efficient portfolios.

Definition 4 *A portfolio p is said to be mean-variance efficient if, for some real numbers a and b, it is a solution of the following optimization problem*

$$\min V(p) \ \ \textit{subject to} \ \ E(p) = a, \ C(p) = b.$$

In order to exhibit the trade-off between mean and variance, we consider the return for taking each additional unit of risk. This is the so-called Sharpe measure of a portfolio. One may want to find the maximal trade-off between mean and variance by formulating costless portfolios. This defines the maximal Sharpe measure δ below.

Definition 5 *Let $\delta = \sup |E(p)|/V^{1/2}(p)$ subject to $C(p) = 0$ and $V(p) \neq 0$.*

We shall also need the following simple concept.

Definition 6 *The asset market is said to be trivial if the expected return function $\mu(t)$ is a constant for λ-almost $t \in T$.*

The following lemma relates the benchmark portfolio M used in the beta asset pricing equation to mean-variance efficient portfolios. Note that the proof of such a result is well known (see, for example, [26]), and we include it here for the sake of completeness.

Lemma 3 *Let M be a portfolio, and ρ and α some real numbers such that a beta asset pricing equation holds, i.e., for λ-almost all $t \in T$, $\mu_t = \rho + \alpha \ cov(x_t, M)$. If $\alpha \neq 0$, then M is mean-variance efficient. In particular, it is mean-variance efficient when the market is not trivial.*

Proof: If $V(M)$ is zero, M is already mean-variance efficient. So we can assume $V(M) \neq 0$. For a portfolio p, since $E(p) = \int_T p(t)\mu(t)d\lambda(t)$, it is easy to obtain from the beta asset pricing equation that $E(p) = \rho C(p) + \alpha \operatorname{cov}(\mathcal{R}_p, \mathcal{R}_M)$ by changing the relevant integrals. In particular, we have $E(M) = \rho C(M) + \alpha V(M)$, which implies that $V(M) = (E(M) - \rho C(M))/\alpha$. Now let p be an arbitrarily given portfolio with mean $E(M)$ and cost $C(M)$. Then the equality $E(p) = \rho C(p) + \alpha \operatorname{cov}(\mathcal{R}_p, \mathcal{R}_M)$ implies that $(E(M) - \rho C(M))/\alpha = \operatorname{cov}(\mathcal{R}_p, \mathcal{R}_M)$. Hence $V(M) = \operatorname{cov}(\mathcal{R}_p, \mathcal{R}_M) \leq \sqrt{V(p)}\sqrt{V(M)}$. Thus $V(M) \leq V(p)$, which means that M is mean-variance efficient. ■

Note that unlike [16, Theorems 2 and 3], a coefficient α is inserted in the beta asset pricing equation in Lemma 3 above. In so far as the existence of a benchmark portfolio is concerned, one can consider αM rather than M, and thus drop α from the equation without any loss of generality. The reason to include α here is to have a sort of linearity among all the portfolios M which satisfies a beta asset pricing equation, and also because one can offer nice interpretations for the coefficients ρ and α. The interpretation in the following corollary presents the usual form of the beta asset pricing equation, where the portfolio M is in fact assumed to be of unit cost.

Corollary 3 *Let M be a portfolio, and ρ and α some real numbers such that for λ-almost all $t \in T$, $\mu_t = \rho + \alpha\ cov(x_t, M)$. Assume $V(M) \neq 0$. Then $\alpha = (E(M) - \rho C(M))/V(M)$. Denote $\beta_t = cov(x_t, M)/V(M)$, the beta of the asset t. Then $\mu_t = \rho + \beta_t(E(M) - \rho C(M))$ for λ-almost all $t \in T$. Moreover, for any portfolio p, we have $E(p) = \rho C(p) + \beta_p(E(M) - \rho C(M))$, where β_p is $cov(p, M)/V(M)$, the beta of the portfolio p. Furthermore the following statements hold.*

(i) If there is a normalized riskless portfolio r, then ρ is the return μ_0 of the portfolio r, and thus $\mu_t - \mu_0 = \beta_t(E(M) - \mu_0 C(M))$ for λ-almost all $t \in T$.

(ii) If there is no riskless asset, then ρ is the expected return of a zero beta portfolio with a unit cost.

Proof: As in the proof of Lemma 3, $E(M) = \rho C(M) + \alpha V(M)$, and which implies that $\alpha = (E(M) - \rho C(M))/V(M)$. It is also obvious that for any portfolio p, we have $E(p) = \alpha C(p) + (E(M) - \rho C(M))\beta_p$. The rest is clear. ■

As indicated in Lemma 3, the portfolio M in the beta asset pricing equation is in general mean-variance efficient. We conclude this subsection by a formal definition that singles out portfolios satisfying the equation.

Definition 7 *A mean-variance efficient portfolio p is said to be a benchmark portfolio if there are real numbers ρ and α such that for λ-almost all $t \in T$, $\mu_t = \rho + \alpha\ cov(x_t, p)$.*

6.2 Linear independence of mean and cost portfolio returns

In the literature on one-period asset pricing, it is often assumed that the mean and cost portfolio returns span a two dimensional space. Here we characterize the case when the two are dependent, and hence the characterization for the independent case follows easily.

Proposition 3 *Let m be a mean portfolio and c a cost portfolio. Then the following are equivalent:*

(i) $\mathcal{R}_m$ and $\mathcal{R}_c$ are linearly dependent;

(ii) there is a real number α such that $\mu(t) = \alpha$ for λ-almost all $t \in T$;

(iii) The following matrix is singular.

$$\begin{pmatrix} (\mathcal{R}_m, \mathcal{R}_m) & (\mathcal{R}_m, \mathcal{R}_c) \\ (\mathcal{R}_c, \mathcal{R}_m) & (\mathcal{R}_c, \mathcal{R}_c) \end{pmatrix} = \begin{pmatrix} E(m) & E(c) \\ E(c) & C(c) \end{pmatrix}.$$

Proof: If $\mathcal{R}_m$ and $\mathcal{R}_c$ are linearly dependent, then there is a real number α such that $\mathcal{R}_m = \alpha\mathcal{R}_c$ since $\mathcal{R}_c$ is never zero. Then, by the definition of mean and cost portfolios, we have for λ-almost all $t \in T$, $\mu_t = (\mathcal{R}_m, x_t) = \alpha(\mathcal{R}_c, x_t) = \alpha$. Hence $(i) \Longrightarrow (ii)$.

For $(ii) \Longrightarrow (i)$, we assume that $\mu(t) \equiv \alpha$ for some real number α. Note that for λ-almost all $t \in T$, $\mu_t = (\mathcal{R}_m, x_t) = \alpha = \alpha(\mathcal{R}_c, x_t)$, and hence $(\mathcal{R}_m - \alpha\mathcal{R}_c, x_t) = 0$ for λ-almost all $t \in T$. By Proposition 2, we have $\mathcal{R}_m - \alpha\mathcal{R}_c = 0$, and hence (i) holds.

Note that $(i) \Longrightarrow (iii)$ is obvious. It remains to see that $(iii) \Longrightarrow (i)$. Assume the matrix in (iii) is singular. Then there is a real number α such that

$$((\mathcal{R}_m, \mathcal{R}_m), (\mathcal{R}_m, \mathcal{R}_c)) = (\alpha(\mathcal{R}_c, \mathcal{R}_m), \alpha(\mathcal{R}_c, \mathcal{R}_c)) .$$

Thus $(\mathcal{R}_m - \alpha\mathcal{R}_c, \mathcal{R}_m) = 0$ and $(\mathcal{R}_m - \alpha\mathcal{R}_c, \mathcal{R}_c) = 0$, which implies that $(\mathcal{R}_m - \alpha\mathcal{R}_c, \mathcal{R}_m - \alpha\mathcal{R}_c) = 0$. Hence $\mathcal{R}_m = \alpha\mathcal{R}_c$. ■

By the above proposition, if mean and cost portfolios exist, then the market is trivial if and only if the two portfolio returns are linearly dependent.

The following simple lemma is useful for the explicit expressions of mean-variance efficient portfolios. Part (i) of the lemma is well known (see, for example, [26]) and part (ii) is also essentially known. We include the lemma for the sake of completeness.

Lemma 4 *Assume that mean and cost portfolios m and c exist and that their returns are linearly independent. Then the following statements are valid.*

(i) A portfolio q is mean-variance efficient if and only if $\mathcal{R}_q$ is in the linear span of $\mathcal{R}_m$ and $\mathcal{R}_c$.

(ii) The portfolio p where

$$p(t) = \frac{(aC(c) - bE(c))\, m(t) + (bE(m) - aE(c))\, c(t)}{E(m)C(c) - (E(c))^2},$$

is mean-variance efficient with mean a and cost b.

Proof: We begin with the proof of (i). For a portfolio q, let $\alpha\mathcal{R}_m + \beta\mathcal{R}_c$ be the projection of $\mathcal{R}_q$ on the space spanned by $\mathcal{R}_m$ and $\mathcal{R}_c$. Define a portfolio u and w by letting $u(t) = \alpha m(t) + \beta c(t)$ and $w(t) = q(t) - u(t)$. Then $\mathcal{R}_w \perp \mathcal{R}_m$ and $\mathcal{R}_w \perp \mathcal{R}_c$, and hence $E(w) = C(w) = 0$, $E(u) = E(q)$ and $C(u) = C(q)$. Since $\mathcal{R}_u \perp \mathcal{R}_w$, we have $V(q) = V(u) + V(w)$. Therefore, the portfolio q is mean-variance efficient if and only if $V(w) = 0$, and thus (i) follows.

For the proof of (ii), note that by (i), we only have to choose real numbers α and β so that the portfolio $p = \alpha m + \beta c$ has mean a and cost b. That is, $\alpha E(m) + \beta E(c) = a$ and $\alpha C(m) + \beta C(c) = b$. Since $C(m) = \int_\Omega \mathcal{R}_c\mathcal{R}_m dP(\omega) = E(c)$, it is equivalent to solve the following equations:

$$\begin{pmatrix} E(m) & E(c) \\ E(c) & C(c) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Since the relevant matrix is nonsingular by Proposition 3, we can obtain that

$$\alpha = \frac{(aC(c) - bE(c))}{E(m)C(c) - (E(c))^2} \text{ and } \beta = \frac{(bE(m) - aE(c))}{E(m)C(c) - (E(c))^2}.$$

We are done. ■

In addition to our standing assumption that the market does not allow gains from exact arbitrage, we assume in the rest of this subsection that both mean and cost portfolios exist leading to a two-dimensional return space. Then, by Theorems 4 and 5, either $\mu_0 \neq 0$ or $h \equiv 0$. By plugging in the formulae for the mean and cost portfolios in Section 4 into the formula in Lemma 4(ii), we obtain immediately all the mean-variance efficient portfolios for these two cases. The formulae for benchmark portfolios and the maximal Sharpe measure δ can also be obtained easily. The results for the case $\mu_0 \neq 0$ are in Proposition 4 and Corollaries 4 and 5. The results for the case $h \equiv 0$ are in Proposition 5 and Corollaries 6 and 7.

We first consider the case where $\mu_0 \neq 0$. Since the cost portfolio exists, Theorem 5(i) says that $\sum_{n=1}^{\infty} \left(\mu_0 s_n - \mu_n\right)^2 / \lambda_n^4 < \infty$. In this case, μ_0 is the riskless interest rate as noted in Theorem 2, and Theorem 4 also shows that a mean portfolio exists. In addition, Proposition 3 indicates that the mean and cost portfolios are independent when the market is not trivial.

Proposition 4 *Let a and b be two arbitrary real numbers. Assume that $\mu_0 \neq 0$, $\mu_s \equiv 0$, and that the asset market is not trivial and cost portfolios exist. Then the portfolio p defined by*

$$p(t) = \frac{\left[a\left(1 + \sum_{n=1}^{\infty} \frac{(\mu_0 s_n - \mu_n)^2}{\lambda_n^2}\right) - b\mu_0\right] m(t) + \left(b\mu_0^2 - a\mu_0\right) c(t)}{\sum_{n=1}^{\infty} \frac{(\mu_0 s_n - \mu_n)^2}{\lambda_n^2}}$$

is mean-variance efficient with mean a and cost b, where $m(t) = h(t)/(\mu_0 h_0)$ is a mean portfolio, and $c(t) = c_0 h(t) + \sum_{n=1}^{\infty} c_n \psi_n(t)$ is a cost portfolio with

$$c_0 = \frac{(1/\mu_0) - \sum_{n=1}^{\infty} c_n \mu_n}{\mu_0 h_0}, \qquad \text{and} \qquad c_n = \frac{\mu_0 s_n - \mu_n}{\mu_0 \lambda_n^2} \quad \text{for all } n \geq 1.$$

Moreover, if q is any mean-variance efficient portfolio with mean a and cost b, then q is the sum of p and a dummy portfolio q_s.

Proof: By Theorems 3 and 4, mean and cost portfolios have the form as given in the statement of the above proposition. It is clear that $E(m) = 1$, $E(c) = 1/\mu_0$, and

$$C(c) = \frac{1}{\mu_0^2} + \sum_{n=1}^{\infty} \lambda_n^2 c_n^2 = \frac{1}{\mu_0^2}\left(1 + \sum_{n=1}^{\infty} \frac{\left(\mu_0 s_n - \mu_n\right)^2}{\lambda_n^2}\right).$$

The rest is clear from Lemmas 1 and 4. ∎

By taking the portfolios described in the previous proposition, we can simply check when p is a benchmark portfolio. The following corollary is then simple to establish. Here we note that by [16, Corollary 3], the existence of a benchmark portfolio is equivalent to $\sum_{n=1}^{\infty} \left(\mu_0 s_n - \mu_n\right)^2 / \lambda_n^4 < \infty$, and hence the assumption on the existence of a cost portfolio is also necessary for the existence of a benchmark portfolio in the case $\mu_0 \neq 0$.

Corollary 4 *Under the hypotheses of Proposition 4, a mean-variance efficient portfolio p is a benchmark portfolio if and only if $E(p) \neq \mu_0 C(p)$, which is also equivalent to $V(p) \neq 0$.*

Proof: Let q be a mean variance efficient portfolio with mean a and cost b in the situation considered in Proposition 4. Then it is easy to check that

$$\begin{aligned}
\operatorname{cov}(x_t, q) &= \frac{(b\mu_0^2 - a\mu_0)\sum_{n=1}^{\infty}\lambda_n^2 c_n \psi_n(t)}{\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2} \\
&= \frac{(b\mu_0 - a)\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)\psi_n(t)}{\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2} \\
&= \frac{(a - b\mu_0)\left(\sum_{n=1}^{\infty}\mu_n \psi_n(t) - \mu_0(1 - h(t))\right)}{\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2} \\
&= \frac{(a - b\mu_0)(\mu(t) - \mu_0)}{\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2},
\end{aligned}$$

and the rest is clear. ■

Note that if the trade-off parameter δ is achieved by some portfolio, then the portfolio must be mean-variance efficient. It is easy to obtain the following corollary by computing the variance of the portfolio p in Proposition 4 for the case $b = 0$.

Corollary 5 *Under the hypotheses of Proposition 4, we have*

$$\delta = \sqrt{\sum_{n=1}^{\infty}\left(\frac{\mu_0 s_n - \mu_n}{\lambda_n}\right)^2}.$$

Proof: For any portfolio q with $V(q) \not\equiv 0$, $E(q) = a$ and $C(q) = 0$, consider the portfolio p in Proposition 4 with $E(p) = a$ and $C(p) = 0$. Then p becomes

$$p(t) = \frac{a\left(1 + \sum_{n=1}^{\infty}\frac{(\mu_0 s_n - \mu_n)^2}{\lambda_n^2}\right)m(t) - a\mu_0 c(t)}{\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2},$$

and hence

$$V(p) = \frac{a^2\mu_0^2\sum_{n=1}^{\infty}\lambda_n^2 c_n^2}{\left(\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2\right)^2} = \frac{a^2}{\sum_{n=1}^{\infty}\left(\frac{\mu_0 s_n - \mu_n}{\lambda_n}\right)^2}.$$

But $|E(p)|/V^{1/2}(p) = \sqrt{\sum_{n=1}^{\infty}\left(\frac{\mu_0 s_n - \mu_n}{\lambda_n}\right)^2}$ for $a \neq 0$. Since p is mean-variance efficient, we know $V(p) \leq V(q)$, and hence, for $a \neq 0$,

$$|E(q)|/V^{1/2}(q) \leq |E(p)|/V^{1/2}(p) = \sqrt{\sum_{n=1}^{\infty}\left(\frac{\mu_0 s_n - \mu_n}{\lambda_n}\right)^2}.$$

If $a = 0$ and $V(q) \not\equiv 0$, we have $|E(q)|/V^{1/2}(q) = 0$, and thus our formula for δ is validated. ■

Next we consider the case $h \equiv 0$. Assume that both mean and cost portfolios exist. By Theorem 4, the existence of a mean portfolio implies that $\sum_{n=1}^{\infty}\mu_n^2/\lambda_n^4 < \infty$. By the proof of

Theorem 5(iv), the existence of a cost portfolio c implies that $\sum_{n=1}^{\infty}(s_n - E(c)\mu_n)^2/\lambda_n^4 < \infty$, and hence $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$. From the convergence of these two series, it is clear that all the series in the following Proposition 5 are convergent. It presents explicit formulae for all mean-variance efficient portfolios. Note that the relevant formulas are more complicated than those in Proposition 4. This is partially due to the fact that there is no normalized riskless portfolio in the market and thus the mean portfolio given here is already quite complicated.

Proposition 5 *Let a and b be arbitrary real numbers. Assume that $h \equiv 0$, $\mu_s \equiv 0$, and that the asset market is not trivial and both mean and cost portfolios exist. Then the portfolio p defined by*

$$\begin{aligned} p(t) &= \frac{\left(a\left[\left(1+\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\right)\sum_{n=1}^{\infty}\frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)^2\right] - b\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right) m(t)}{\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\sum_{n=1}^{\infty}\frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)^2} \\ &\quad + \frac{\left(b\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2} - a\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right) c(t)}{\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\sum_{n=1}^{\infty}\frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)^2} \end{aligned}$$

is mean-variance efficient with mean a and cost b, where $m(t) = \sum_{n=1}^{\infty} m_n\psi_n(t)$ and $c(t) = \sum_{n=1}^{\infty} c_n\psi_n(t)$ are the mean and cost portfolios respectively with

$$m_n = \left(1+\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\right)^{-1}\frac{\mu_n}{\lambda_n^2},\ c_n = \frac{s_n - \gamma\mu_n}{\lambda_n^2} \text{ and } \gamma = \frac{\sum_{n=1}^{\infty}(\mu_n s_n \,/\, \lambda_n^2)}{1+\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2)}.$$

Moreover, if q is any mean-variance efficient portfolio with mean a and cost b, then q is the sum of p and a dummy portfolio q_s.

Proof: By Theorems 3 and 4, the given m and c are indeed mean and cost portfolios with $E(m) = (\sum_{n=1}^{\infty}\mu_n^2/\lambda_n^2)/(1+\sum_{n=1}^{\infty}\mu_n^2/\lambda_n^2)$ and $E(c) = \gamma$. We also have

$$C(c) = \sum_{n=1}^{\infty} c_n s_n = \sum_{n=1}^{\infty}\frac{s_n^2}{\lambda_n^2} - \gamma\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2} = \frac{\left(1+\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\right)\sum_{n=1}^{\infty}\frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)^2}{1+\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}}.$$

Hence

$$E(m)C(c) - (E(c))^2 = \frac{\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\sum_{n=1}^{\infty}\frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)^2}{1+\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}}.$$

Note that the fact that the market is not trivial implies that $E(m)C(c) - (E(c))^2$ is not zero. The rest is clear from Lemmas 1 and 4. ■

The following Corollary 6 characterizes the benchmark portfolios.

Corollary 6 *Under the hypotheses of Proposition 5, a mean variance efficient portfolio p is a benchmark portfolio if and only if $E(p) \neq C(p)(\tau/\beta)$, where $\alpha = \sum_{n=1}^{\infty}\mu_n^2/\lambda_n^2$, $\beta = \sum_{n=1}^{\infty} s_n^2/\lambda_n^2$, and $\tau = \sum_{n=1}^{\infty}\mu_n s_n/\lambda_n^2$.*

Proof: By using the notation introduced in the statement of the corollary, the portfolio defined by

$$p(t) = \frac{(a[(1+\alpha)\beta - \tau^2] - b\tau)\, m(t) + (b\alpha - a\tau c(t)}{\alpha\beta - \tau^2}$$

has mean a and cost b. Then it is easy to obtain that

$$\mathcal{R}_p = \frac{(a[(1+\alpha)\beta - \tau^2] - b\tau)\, \mathcal{R}_m + (b\alpha - a\tau)\mathcal{R}_c}{\alpha\beta - \tau^2}.$$

Note that $\mathrm{cov}(x_t, p) = (x_t, \mathcal{R}_p) - E(x_t)E(p)$, and hence

$$\mathrm{cov}(x_t, p) = \frac{(a[(1+\alpha)\beta - \tau^2] - b\tau)\, \mu(t) + b\alpha - a\tau}{\alpha\beta - \tau^2} - a\mu(t) = \frac{(a\beta - b\tau)\mu(t) + b\alpha - a\tau}{\alpha\beta - \tau^2},$$

where the fact that m and c are the mean and cost portfolios is used. Hence p is a benchmark portfolio if and only if $E(p) \neq C(p)(\tau/\beta)$. ∎

The last corollary of this subsection provides an exact formula for the maximal Sharpe measure δ.

Corollary 7 *Under the hypotheses of Proposition 5, we can obtain*

$$\delta = \sqrt{\frac{\sum_{n=1}^{\infty} \frac{\mu_n^2}{\lambda_n^2} \sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2}\right)^2}{\sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^2}}}.$$

Proof: As in the proof of Corollary 5, our δ is equal to $|E(p)|/V^{1/2}(p)$ for p in Proposition 5 with $E(p) = 1$ and $C(p) = 0$. We use the notation of Corollary 6. Hence

$$p(t) = \frac{[(1+\alpha)\beta - \tau^2]\, m(t) - \tau\, c(t)}{\alpha\beta - \tau^2}.$$

Thus

$$\begin{aligned}
V(p) &= \frac{\sum_{n=1}^{\infty} \lambda_n^2 \left([(1+\alpha)\beta - \tau^2]\, m_n - \tau c_n\right)^2}{(\alpha\beta - \tau^2)^2} \\
&= \frac{\sum_{n=1}^{\infty} \lambda_n^2 \left([(1+\alpha)\beta - \tau^2] \frac{1}{(1+\alpha)} \frac{\mu_n}{\lambda_n^2} - \tau \frac{s_n}{\lambda_n^2} + \tau^2 \frac{1}{1+\alpha} \frac{\mu_n}{\lambda_n^2}\right)^2}{(\alpha\beta - \tau^2)^2} \\
&= \frac{\sum_{n=1}^{\infty} \left(\beta \frac{\mu_n}{\lambda_n} - \tau \frac{s_n}{\lambda_n}\right)^2}{(\alpha\beta - \tau^2)^2} \\
&= \frac{\beta^2 \sum_{n=1}^{\infty} (\frac{\mu_n}{\lambda_n})^2 - 2\beta\tau \sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2} + \tau^2 \sum_{n=1}^{\infty} (\frac{s_n}{\lambda_n})^2}{(\alpha\beta - \tau^2)^2} \\
&= \frac{\beta^2\alpha - 2\beta\tau^2 + \tau^2\beta}{(\alpha\beta - \tau^2)^2} = \frac{\beta}{\alpha\beta - \tau^2}.
\end{aligned}$$

Therefore $\delta = \sqrt{(\alpha\beta - \tau^2)/\beta)}$, and we are done. ∎

6.3 The case of no cost portfolios

In this subsection, we consider a situation where a mean portfolio exists, and $\mu_0 = 0$ and $h \not\equiv 0$. By Theorem 5(ii), cost portfolios do not exist. However, we can still find mean-variance efficient portfolios with any given means and costs when there is no possibility of gains from exact arbitrage and when the mean function is not the constant zero. Since a mean portfolio exists, Theorem 4 implies that $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^4 < \infty$. We first prove a version of Lemma 4.

Lemma 5 *Assume that $\mu_0 = 0$, $h \not\equiv 0$, $\mu_s \equiv 0$, $\mu \not\equiv 0$, and there is a mean portfolio m. Then a portfolio q is mean-variance efficient if and only if $\mathcal{R}_q$ is a multiple of $\mathcal{R}_m$.*

Proof: For a portfolio q with mean a and cost b, define portfolios

$$u = \left(\frac{a}{E(m)}\right) m + \left(\frac{b}{h_0} - \frac{ac(m)}{h_0 E(m)}\right) h$$

and $w = q - u$. Note that, by the second part of Theorem 4, the assumptions of the lemma imply that $E(m) \neq 0$, and hence the portfolios are well defined. It is easy to check that $C(u) = b$ and $\mathcal{R}_u$ is $(a/E(m))\mathcal{R}_m = \frac{(\mathcal{R}_m, \mathcal{R}_q)}{(\mathcal{R}_m, \mathcal{R}_m)}\mathcal{R}_m$, the projection of $\mathcal{R}_q$ on the linear space spanned by $\mathcal{R}_m$, and hence $\mathcal{R}_w \perp \mathcal{R}_m$. Thus $V(q) = V(u) + V(w)$. Therefore, the portfolio q is mean-variance efficient if and only if $V(w) = 0$, and thus the lemma follows. ■

The following proposition characterizes all the mean-variance portfolios in this setting.

Proposition 6 *Let a and b be arbitrary real numbers. Assume that $\mu_0 = 0$, $h \not\equiv 0$, $\mu_s \equiv 0$, $\mu \not\equiv 0$, and that a mean portfolio exists. Then the portfolio p defined by*

$$p(t) = \left[\frac{b}{h_0} - \frac{a\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}}{h_0 \sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2)}\right] h(t) + a\frac{\sum_{n=1}^{\infty}\frac{\mu_n}{\lambda_n^2}\psi_n(t)}{\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2)}$$

is mean-variance efficient with mean a and cost b. Moreover, if q is any mean-variance efficient portfolio with mean a and cost b, then q is the sum of p and a dummy portfolio q_s.

Proof: As shown in Lemma 5, the portfolio

$$p = \left(\frac{a}{E(m)}\right) m + \left(\frac{b}{h_0} - \frac{ac(m)}{h_0 E(m)}\right) h$$

is mean-variance efficient. Here we choose m to be the mean portfolio defined in the last paragraph of the proof of Theorem 4. Then

$$E(m) = \frac{\sum_{n=1}^{\infty}\mu_n^2/\lambda_n^2}{1 + \sum_{n=1}^{\infty}\mu_n^2/\lambda_n^2} \text{ and } C(m) = \sum_{n=1}^{\infty} m_n s_n = \left(1 + \sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\right)^{-1} \sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2},$$

and hence we obtain the formula for p as described.

Next, let $q = q_0 h + \sum_{n=1}^{\infty} q_n \psi_n + q_s$ such that q is mean-variance efficient with mean a and cost b. Then we must have $\mathcal{R}_q = \mathcal{R}_p$, and hence $\mathcal{R}_{q-p} = 0$. This implies that $q_n - p_n = 0$ for all $n \geq 1$. Now

$$C(q) = q_0 h_0 + \sum_{n=1}^{\infty} q_n s_n = q_0 h_0 + \sum_{n=1}^{\infty} p_n s_n$$

and $C(p) = p_0 h_0 + \sum_{n=1}^{\infty} p_n s_n$. By $C(p) = C(q)$ and $h_0 \neq 0$, we obtain $p_0 = q_0$. Therefore $q = p + q_s$. ■

Let q be a mean variance efficient portfolio with mean a and cost b in the situation considered in Proposition 6. Then it is easy to check that $\mathrm{cov}(x_t, q) = a\mu(t)/(\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2)$. Hence the following corollary is clear.

Corollary 8 *Under the hypotheses of Proposition 6, a mean-variance efficient portfolio q is a benchmark portfolio if and only if $E(q) \neq 0$, which is equivalent to $V(q) \neq 0$.*

Next, let p be a mean-variance efficient portfolio with mean a and cost b as in Proposition 6. Then it is easy to check that $V(p) = a^2 / \sum_{n=1}^{\infty} (\mu_n^2/\lambda_n^2)$, and hence we can obtain a formula for the maximal Sharpe measure δ as follows.

Corollary 9 *Under the hypotheses of Proposition 6, we obtain $\delta = \sqrt{\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2}$.*

6.4 The linear dependence of mean and cost portfolio returns

Finally, we consider the case when the market is trivial. By Proposition 3, the mean and cost portfolios are linearly dependent if they exist. In this case, there is no meaningful trade-off between mean and variance, and taking more risk without spending more does not lead to a higher return. The expected return is simply linearly dependent on the cost with a fixed coefficient. Thus, it is no longer true that for every pair of given amount of return and cost, there is a mean-variance efficient portfolio corresponding to it. The following proposition describes all the relevant mean-variance efficient portfolios.

Proposition 7 *Let b be a real number. Assume that the asset market is trivial, with the expected return function μ being equal to a constant α.*

(i) If $h \not\equiv 0$, then the variance of the portfolio p defined by $p(t) = b\, h(t)/h_0$ is zero with mean αb and cost b, and hence is also mean-variance efficient.

(ii) If $h \equiv 0$ and $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$, then the portfolio p defined by

$$p(t) = b \frac{\sum_{n=1}^{\infty} \frac{s_n}{\lambda_n^2} \psi_n(t)}{\sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^2}}$$

is mean-variance efficient with mean αb and cost b.

Moreover, if q is any mean-variance efficient portfolio with cost b, then q is the sum of the portfolio p in (i) or (ii), and a dummy portfolio q_s.

Proof: We begin with the proof of (i). Since $\mu \equiv \alpha$, it is clear that $E(q) = \alpha C(q)$ for any portfolio q. We can only allow the value of cost to be a variable. The portfolio $p = b\,h/h_0$ has mean αb and cost b. Since its variance is zero, it must be mean-variance efficient. Next, let $q = q_0 h + \sum_{n=1}^{\infty} q_n \psi_n + q_s$ be a mean-variance efficient portfolio with cost b. Then $\mathcal{R}_q$ must be constant, which implies that $q_n = 0$ for all $n \geq 1$. Since $C(q) = q_0 h_0 = b$, we can obtain $q_0 = b/h_0$, and hence $q = p + q_s$.

For the proof of (ii), note that the cost portfolio exists in this case. Since we do not have to consider a constraint involving the mean, as in Lemma 5, we can obtain that a portfolio q is mean-variance efficient if and only if $\mathcal{R}_q$ is a multiple of $\mathcal{R}_c$. By Theorem 5(iv) and the fact that $\mu = \alpha$, we obtain a cost portfolio c with

$$c(t) = \left(1 + \alpha^2 \sum_{n=1}^{\infty} s_n^2\right)^{-1} \sum_{n=1}^{\infty} \frac{s_n}{\lambda_n^2} \psi_n(t).$$

The portfolio p defined in (ii) is a multiple of c with cost b, and hence also mean-variance efficient.

Next, if q is a is mean-variance efficient portfolio with cost b, then $\mathcal{R}_{q-p}$ must be zero. By Lemma 1(ii), $q - p$ must be a dummy portfolio. ■

When the asset market is trivial, it is clear that every mean-variance efficient portfolio is a benchmark portfolio. One can simply take the coefficient in front of the covariance in Definition 7 to be zero. Note that here we already use α to be the constant expected return of all the assets. As noted earlier, there is no meaningful trade-off between mean and variance for a trivial market. However, we can define a trade-off between cost and variance.

Definition 8 *Let* $\delta_c = \inf V^{1/2}(p)/C(p)$ *subject to* $C(p) \neq 0$.

Note that δ_c measures the minimum risk for each additional cost. As in the previous cases, we can obtain the following simple corollary, whose proof will be omitted.

Corollary 10 *Assume the market is trivial. Then (i) if* $h \not\equiv 0$, *then* $\delta_c = 0$; *(ii) if* $h \equiv 0$ *and* $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$, *then* $\delta_c = 1/\left(\sqrt{\sum_{n=1}^{\infty}(s_n^2/\lambda_n^2)}\right)$.

7 Concluding Remarks

The results obtained in this paper rely crucially on two results developed in [28] and [29]. The first is an exact law of large numbers that allows the complete removal of idiosyncratic risk in a portfolio. This is the reason why all of the formulae for the variety of portfolios presented above only contain factor risks. The second result is a generalization of the Karhunen-Loéve expansion theorem to the hyperfinite setting. It is the bi-orthogonality property in such an expansion that renders possible the computation of various portfolios.[17] The underlying measure-theoretic framework uses Loeb measures.[18] As emphasized above, the advantage of the Loeb measure

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[17] These two results form the backbone of the portmanteau theorem in Section 2 above.

[18] For other ongoing work in financial economics based on the Loeb space, see [3, 4] and [22].

framework is that it provides a richer product measure space for a viable study of a continuum of random variables with low intercorrelation.[19]

We conclude this paper with two remarks pertaining to Loeb spaces in the substantive context of this paper. First, Hilbert spaces based on atomless Loeb probability spaces, as used here and unlike those based on the unit Lebesgue interval, are non-separable. This shows up, perhaps most dramatically, in Proposition 1, which guarantees the existence of financial markets allowing for the possibility of gains from exact arbitrage despite the presence of a general infinite-dimensional factor subspace. Second, in the introduction to this paper, we have used the (asymptotic) APT as a benchmark against which to present the (exact) APT. However, the reader should note that many of our results, based as they are on a Loeb space, can be asymptotically implemented to a setting with a large but finite number of assets; see [16, Section 3.2]. Such an implementation is conceptually distinct from the (asymptotic) APT and involves analogues of the key parameters (s_n, μ_n, h, μ_0) for finite markets. We leave this as an exercise for the interested reader.

8 Appendix

The proof of Proposition 1 hinges on the following lemma on infinite matrices, which should be well known in the literature on Hilbert spaces. We include it for the sake of completeness.

Lemma 6 *Let $\alpha_0, \alpha_1, \alpha_2, \cdots$ be a complete orthonormal basis for the Hilbert space ℓ_2 of sequences of real numbers. For each $j = 0, 1, 2, \cdots$, let $\alpha_j = \{a_{ij}\}_{i=0}^{\infty}$. If the sequence α_j is viewed as the $j+1$-th column vector of an infinite matrix, then the row vectors are also orthonormal in ℓ_2.*

Proof: Let $\sum_{l=0}^{\infty} a_{il} a_{jl} = b_{ij}$. We have to show that $b_{ij} = \delta_{ij}$, which is one when $i = j$ and zero otherwise.

$$\begin{aligned}\sum_{i=0}^{\infty} a_{ik} b_{ij} &= \sum_{i=0}^{\infty} a_{ik} \left(\sum_{l=0}^{\infty} a_{il} a_{jl} \right) = \sum_{l=0}^{\infty} a_{jl} \left(\sum_{i=0}^{\infty} a_{ik} a_{il} \right) \\ &= \sum_{l=0}^{\infty} a_{jl} \delta_{kl} = a_{jk} = \sum_{i=0}^{\infty} a_{ik} \delta_{ij}.\end{aligned}$$

This implies that $\sum_{i=0}^{\infty} a_{ik}(b_{ij} - \delta_{ij}) = 0$ for all $k = 0, 1, 2, \cdots$. Since the α_k's form a complete orthonormal basis for ℓ_2, this implies $b_{ij} = \delta_{ij}$ for all $i = 0, 1, 2, \cdots$, and all $j = 0, 1, 2, \cdots$. ■

We can now present

Proof of Proposition 1: Let $\alpha_0 = (s_0, s_1, \cdots)'$ be an infinite column vector such that $s_0^2 = 1 - \sum_{n=1}^{\infty} s_n^2$. Find column vectors $\alpha_1, \alpha_2, \cdots$ so that $\alpha_0, \alpha_1, \alpha_2, \cdots$ is a complete orthonormal basis for ℓ_2. For any $j = 0, 1, 2, \cdots$, let $\alpha_j = \{a_{ij}\}_{i=0}^{\infty}$. By Lemma 6, the sequences $\beta_i = \{a_{ij}\}_{j=0}^{\infty}$ for any $i = 0, 1, 2, \cdots$ is orthonormal in ℓ_2.

Now consider a orthonormal sequence of functions $\gamma_i, i = 0, 1, \cdots$ on T such that γ_0 is the unit function, and all the other functions have mean zero. It is always possible to find such a sequence of functions on T since λ is atomless. Define for each $i = 1, 2, \cdots, \psi_i = \sum_{j=0}^{\infty} a_{ij} \gamma_j$. Certainly the sequence $\{\psi_i\}_{i=1}^{\infty}$ is orthonormal. Also, for any $i = 1, 2, \cdots, \int_T \psi_i d\lambda = a_{i0} \int_T \gamma_0 d\lambda = a_{i0} = s_i$.

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[19] It is proved in [31] that almost all sample functions are *not* Lebesgue measurable in a setting where a continuum of random variables are indexed by points in the unit interval; see also [14] and its references.

We now pick an orthogonal sequence $\{\varphi_n\}_{n=1}^{\infty}$ of random variables on Ω with mean zero and variance one. It is always possible to find such a sequence of random variables on Ω since P is atomless.

Next, we invoke [29, Theorem 6.2] to assert the existence of a process e such that the random variables e_t are almost i.i.d. with the almost common distribution ν. We appeal to Theorem A(iv) to claim that for P-almost all $\omega \in \Omega$, $\int_T e_\omega(t) d\lambda(t) = 0$ and $\int_T \psi_n(t) e_\omega(t) d\lambda(t) = 0$ for all $n \geq 1$. We can now invoke [29, Theorem 7.16] to assert that the sample functions $\omega \in \Omega$, e_ω are almost i.i.d. and the almost common distribution must be ν (by the exact law of large numbers in [28] and [29]). Pick one such ω such that e_ω has distribution ν and $e_\omega \in (\mathcal{L} \vee \{1\})^{\perp}$; define $\mu_s = \sqrt{a} e_\omega$.

The proof is finished. ■

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Market Parameters[1]	**Types of Portfolios**			
	Normalized Riskless[2]	**Factor**[3]	**Mean**[4]	**Cost**[5]
$h \equiv 0, \mu_s \not\equiv 0$	Remark 1(i)	Remark 4(ii)	Remarks 6(i), 9(i)	Remark 9
$h \equiv 0, \mu_s \equiv 0$	Remark 1(i)	Remark 3	Remarks 5, 6(ii), 10	Remark 10
$h \not\equiv 0, \mu_0 = 0, \mu_s \equiv 0$	Remark 1(ii)	Remark 3	Remarks 5, 6(ii), 8(ii)	Remark 8(ii)
$h \not\equiv 0, \mu_0 = 0, \mu_s \not\equiv 0$	Remark 2(ii)	Remark 4(ii)	Remarks 6(i), 8(i)	Remark 8(i)
$h \not\equiv 0, \mu_0 \neq 0, \mu_s \equiv 0$	Remark 2(i)	Remark 4(i)	Remark 6(i)	Remark 7(ii, iii)
$h \not\equiv 0, \mu_0 \neq 0, \mu_s \not\equiv 0$	Remark 2(iii)	Remark 4(ii)	Remark 6(i)	Remark 7(i)

1. Recall that $\mu_0 = 0$ when $h \equiv 0$ and that the converse is false.
2. This column illustrates Theorem 2: $\exists$ normalized riskless portfolio $\iff (h \not\equiv 0$ and $(\mu_0 \neq 0$ or $\mu_s \neq 0))$.
3. This column illustrates Theorem 3: All factors are portfolio returns $\iff (\mu_0 \neq 0$ or $\mu \equiv 0$ or $\mu_s \neq 0)$.
   Remark 4(iii) concerns the case $\mu \equiv 0$ not tabulated here.
4. This column illustrates Theorem 4: (i) $\exists$ mean portfolio if $\mu_0 \neq 0$ or $\mu_s \neq 0$.
   (ii) If $(\mu_0 = 0$ and $\mu_s = 0), \exists$ mean portfolio $\iff \sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^4) < \infty$.
5. This column illustrates Theorem 5: (i) If $(\mu_0 \neq 0) \exists$ cost portfolio $\iff (\mu_s \equiv 0$ and $\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^4 < \infty)$.
   (ii) If $(\mu_0 = 0$ and $h \not\equiv 0)$, $\nexists$ cost portfolio.
   (iii) If $(h \equiv 0$ and $\mu_s \not\equiv 0), \exists$ cost portfolio $\iff \sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$.
   (iv) If $(h \equiv 0, \mu_s \equiv 0,$ and $\sum_{n=1}^{\infty}(s_n^2 + \mu_n^2)/\lambda_n^2 < \infty)$,
   $\exists$ cost portfolio $\iff \sum_{n=1}^{\infty}(s_n - b\mu_n)^2/\lambda_n^4 < \infty$,
   where $b = (\sum_{n=1}^{\infty}(\mu_n s_n/\lambda_n^2))/(1 + \sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2))$.

Table 1: Existence and Non-existence of Different Kinds of Portfolios under Varying Market Parameters