Math Facts Useful for Graduate Macroeconomics

http://econ.jhu.edu/people/ccarroll/public/lecturenotes

The following collection of facts is useful in many macroeconomic models. No proof is offered for some facts because the derivations are standard elements of prerequisite mathematics or microeconomics classes; this handout is offered as an *aide memoire* and for reference purposes.

1 Utility Functions

1.1 [CRRALim]

Fact 1

$$\lim_{\rho\to 1}\left(\frac{c^{1-\rho}}{1-\rho}\right)=\log c \tag{1}$$

To see this note that we can equivalently define

$$\mathbf{u}(c) \;=\; \int_0^c \chi^{-\rho}d\chi \tag{2}$$

and note that this means that if $\rho = 1$ then

$$\mathbf{u}'(c) \;=\; c^{-\rho} \tag{3}$$

but $\int(1/c) = \log c$.

2 Geometric Series

2.1 [FinSum]

Fact 2

$$\sum_{i=0}^{T}\gamma^i=\left(\frac{1-\gamma^{T+1}}{1-\gamma}\right) \tag{4}$$

2.2 [InfSum]

Fact 3 *If* $0 < \gamma < 1$, *then*

$$\sum_{i=0}^{\infty} \gamma^i = \left(\frac{1}{1-\gamma}\right) \tag{5}$$

3 'Small' Number Approximations

3.1 [TaylorOne]

Fact 4 *For* ϵ *near zero ('small'), a first order Taylor expansion of* $(1+\epsilon)^\zeta$ *around 1 yields*

$$(1+\epsilon)^\zeta \approx 1 + \epsilon\zeta \tag{6}$$

3.2 [TaylorTwo]

Fact 5 *For* ϵ *near zero ('small'), a second order Taylor expansion of* $(1+\epsilon)^\zeta$ *around 1 yields*

$$(1+\epsilon)^\zeta \approx 1 + \zeta\epsilon + \epsilon^2\zeta(\zeta-1)/2 \tag{7}$$

$$= 1 + \left(1 + \left(\frac{\zeta-1}{2}\right)\epsilon\right)\zeta\epsilon \tag{8}$$

3.3 [LogEps]

Fact 6 *For* ϵ *near zero ('small'),*

$$\log(1+\epsilon) \approx \epsilon \tag{9}$$

3.4 [ExpEps]

Fact 7 *For* ϵ *near zero ('small'),*

$$(1+\epsilon) \approx e^\epsilon \tag{10}$$

3.5 [OverPlus]

Fact 8 *For ϵ near zero ('small'),*

$$1/(1+\epsilon) \approx 1-\epsilon \tag{11}$$

3.6 [MultPlus]

Fact 9 *For ϵ and ζ near zero ('small'),*

$$(1+\epsilon)(1+\zeta) \approx 1+\epsilon+\zeta \tag{12}$$

3.7 [ExpPlus]

Fact 10 *For real numbers ϵ and ζ*

$$\exp(\zeta)\exp(\epsilon) = \exp(\zeta+\epsilon) \tag{13}$$

4 Statistical Facts

4.1 [ELogNorm]

Fact 11 *If from the viewpoint of period t the stochastic variable Z_{t+1} is lognormally distributed with mean $\bar{z}$ and variance σ_z^2 (Defining $z_{t+1} = \log Z_{t+1}$, write this as $z_{t+1} \sim \mathcal{N}(\bar{z}, \sigma_z^2)$), then*

$$\mathbb{E}_t[e^{z_{t+1}}] = e^{\bar{z}+\sigma_z^2/2} \tag{14}$$

4.2 [LogELogNorm]

Fact 12 *If Z_{t+1} is lognormally distributed as in the prior fact, then*

$$\begin{aligned} \log \mathbb{E}_t[Z_{t+1}] &= \mathbb{E}_t[\log Z_{t+1}] + \sigma_z^2/2 && (15) \\ &= \bar{z} + \sigma_z^2/2 && (16) \end{aligned}$$

which follows from taking the log of both sides of (14).

4.3 [NormTimes]

Fact 13 *If* $z \sim \mathcal{N}(\bar{z}, \sigma_z^2)$*, then*

$$\gamma z \sim \mathcal{N}(\gamma \bar{z}, \gamma^2 \sigma_z^2) \tag{17}$$

4.4 [SmallSmallZero]

Fact 14 *If* ϵ *is small and* ζ *is small then* $\epsilon\zeta$ *can be approximated by zero.*

5 Other Facts

5.1 [EulersTheorem]

Fact 15 *If* $Y = F(K, L)$ *is a constant returns to scale production function, then*

$$Y = F_K K + F_L L, \tag{18}$$

and if this production function characterizes output in a perfectly competitive economy then F_K *is the interest factor and* F_L *is the wage rate.*